A-Level Further Maths Transition Work

Why study A Level Further Maths?

Studying mathematics develops powers of logical thinking, analysis and problem solving. You will be encouraged to understand mathematics and mathematical processes in a way that promotes confidence and fosters enjoyment of the subject. Further Maths is a challenging and stimulating course that will introduce you to new topic areas, including Decision Maths and Pure Maths topics such as Complex numbers and Matrices. The course builds on the skills and topics you study as part of A Level Maths enabling you to develop and extend your understanding of these topics which should help you to improve your A Level Maths grade.

Studying A Level Further Maths:

- will support the study of other A levels including A Level Maths
- develop key employability skills such as problem-solving, logical reasoning, communication and resilience
- is excellent preparation for a wide range of university courses and careers and is seen by many universities as extremely desirable
- leads to versatile qualifications that are well-respected by employers

A-Level Further Maths at a glance

The subject content is divided into three areas: Core Pure Mathematics, Further Mechanics and Decision Maths. You will be expected to be able to use your knowledge to reason mathematically and solve problems both within mathematics and in context. Any of the content may be required in problem solving, modelling and reasoning tasks. The course assessments will require you to develop and demonstrate the ability to draw together different areas of your knowledge, skills and understanding from both the Maths and Further Maths course.

How it is assessed

In your final examination you will have to sit four papers each lasting 1.5 hours and worth 25% of the qualification. All questions are compulsory and there are 75 marks in total.

Paper 1: Core Pure Mathematics 1

Paper 2: Core Pure Mathematics 2

Paper 3: Further pure Mathematics 1

Paper 4: Decision Mathematics 1

The purpose of the transition work

This transition work builds on the work you will be doing in the transition work for A Level Maths. The focus is on consolidating key skills and knowledge, as opposed to learning new material so that you are really fluent in these different topic areas. Some of the tasks focus on key algebra skills but there are also tasks linked to graphs and skills needed in geometry. All of the skills included in this pack will be tested at A-level Maths as well as being skills you need to be able to solve the type of problems you will meet in the Further Maths course, involving more complex mathematics.

To achieve this fluency you need to:

- Read and make notes from the examples, taking partiuclar notice of how to set the answers out. At A-level you must be able to show your methods clearly and know the key details to include – some questions will expect detailed reasoning.
- Work through the practice questions including clear methods.
- Check your answers and annotate them with any key points you need to remember or queries that you need to raise with your teachers in September.

You will have a timed assessment to do on this work in the first two weeks back to check your understanding and skills, in which you should be aiming to get at least 75%. This is an opportunity to check that this course is the right pathway for you.

Transition Task

- Task 1 - Completing the square
- Task 2 - Solving simultaneous equations graphically
- Task 3 - Sketching cubic and reciprocal graphs
- Task 4 - Translating graphs
- Task 5 - Stretching graphs
- Task 6 - Straight line graphs
- Task 7 - Parallel and perpendicular lines
- Task 8 - Pythagoras' theorem
- Task 9 - Circle theorems
- Task 10 - Trigonometry in right-angled triangles
- Task 11 - The cosine rule
- Task 12 - The sine rule
- Task 13 - Areas of triangles
- Task 14 - Rearranging equations

By the end of this task you will:

- have gained knowledge and understanding of completing the square
- be able to write a quadratic expression in the form $p(x + q)^2 + r$

Task 1:

Completing the square

A LEVEL LINKS Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$ $= (x + 3)^2 - 9 - 2$ $= (x + 3)^2 - 11$	**1** Write $x^2 + bx + c$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$ **2** Simplify

Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$	**1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$
$= 2\left(x^2 - \frac{5}{2}x\right) + 1$	**2** Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$
$= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$	
$= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1$	**3** Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2
$= 2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8}$	**4** Simplify

Practice

1 Write the following quadratic expressions in the form $(x + p)^2 + q$

a $x^2 + 4x + 3$ **b** $x^2 - 10x - 3$

c $x^2 - 8x$ **d** $x^2 + 6x$

e $x^2 - 2x + 7$ **f** $x^2 + 3x - 2$

2 Write the following quadratic expressions in the form $p(x + q)^2 + r$

a $2x^2 - 8x - 16$ **b** $4x^2 - 8x - 16$

c $3x^2 + 12x - 9$ **d** $2x^2 + 6x - 8$

3 Complete the square.

a $2x^2 + 3x + 6$ **b** $3x^2 - 2x$

c $5x^2 + 3x$ **d** $3x^2 + 5x + 3$

Extend

4 Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.

Answers

1 **a** $(x + 2)^2 - 1$ **b** $(x - 5)^2 - 28$

c $(x - 4)^2 - 16$ **d** $(x + 3)^2 - 9$

e $(x - 1)^2 + 6$ **f** $\left(x + \frac{3}{2}\right)^2 - \frac{17}{4}$

2 **a** $2(x - 2)^2 - 24$ **b** $4(x - 1)^2 - 20$

c $3(x + 2)^2 - 21$ **d** $2\left(x + \frac{3}{2}\right)^2 - \frac{25}{2}$

3 **a** $2\left(x + \frac{3}{4}\right)^2 + \frac{39}{8}$ **b** $3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3}$

c $5\left(x + \frac{3}{10}\right)^2 - \frac{9}{20}$ **d** $3\left(x + \frac{5}{6}\right)^2 + \frac{11}{12}$

4 $(5x + 3)^2 + 3$

By the end of this task you will:

- have gained knowledge and understanding of using graphs to solve simultaneous equations
- be able to solve simultaneous equations graphically

Task 2:

Solving simultaneous equations graphically

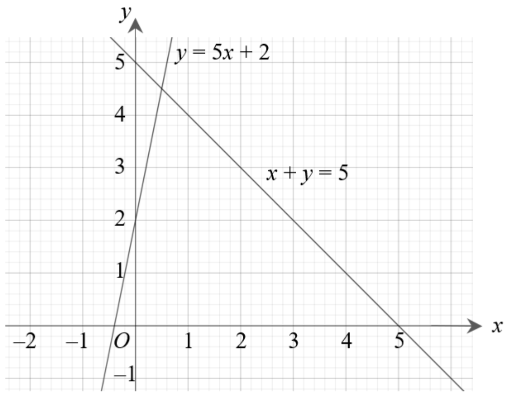
A LEVEL LINKS
Equations – quadratic/linear simultaneous

Key points

- You can solve any pair of simultaneous equations by drawing the graph of both equations and finding the point/points of intersection.

Examples

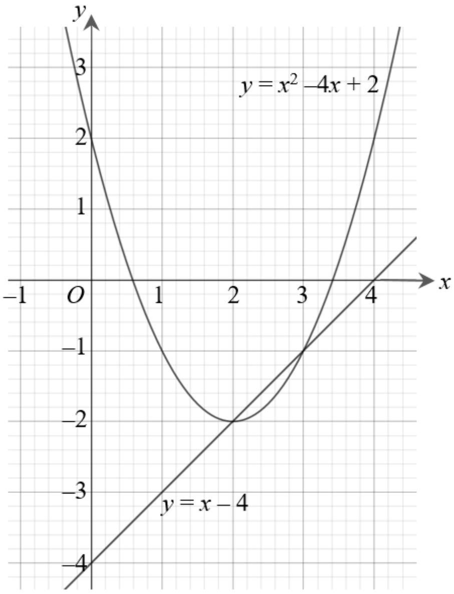
Example 1 Solve the simultaneous equations $y = 5x + 2$ and $x + y = 5$ graphically.

$y = 5 - x$	**1** Rearrange the equation $x + y = 5$ to make y the subject.
$y = 5 - x$ has gradient –1 and y-intercept 5. $y = 5x + 2$ has gradient 5 and y-intercept 2.  	**2** Plot both graphs on the same grid using the gradients and y-intercepts.
Lines intersect at $x = 0.5$, $y = 4.5$	**3** The solutions of the simultaneous equations are the point of intersection.
Check: First equation $y = 5x + 2$: $4.5 = 5 \times 0.5 + 2$ YES Second equation $x + y = 5$: $0.5 + 4.5 = 5$ YES	**4** Check your solutions by substituting the values into both equations.

Example 2 Solve the simultaneous equations $y = x - 4$ and $y = x^2 - 4x + 2$ graphically.

x	0	1	2	3	4
y	2	–1	–2	–1	2

1 Construct a table of values and calculate the points for the quadratic equation.



2 Plot the graph.

3 Plot the linear graph on the same grid using the gradient and y-intercept.
$y = x - 4$ has gradient 1 and y-intercept –4.

The line and curve intersect at
$x = 3, y = -1$ and $x = 2, y = -2$

4 The solutions of the simultaneous equations are the points of intersection.

Check:
First equation $y = x - 4$:
$-1 = 3 - 4$ YES
$-2 = 2 - 4$ YES
Second equation $y = x^2 - 4x + 2$:
$-1 = 3^2 - 4 \times 3 + 2$ YES
$-2 = 2^2 - 4 \times 2 + 2$ YES

5 Check your solutions by substituting the values into both equations.

Practice

1 Solve these pairs of simultaneous equations graphically.

a $y = 3x - 1$ and $y = x + 3$

b $y = x - 5$ and $y = 7 - 5x$

c $y = 3x + 4$ and $y = 2 - x$

2 Solve these pairs of simultaneous equations graphically.

a $x + y = 0$ and $y = 2x + 6$

b $4x + 2y = 3$ and $y = 3x - 1$

Hint

Rearrange the equation to make y the subject.

c $2x + y + 4 = 0$ and $2y = 3x - 1$

3 Solve these pairs of simultaneous equations graphically.

a $y = x - 1$ and $y = x^2 - 4x + 3$

b $y = 1 - 3x$ and $y = x^2 - 3x - 3$

c $y = 3 - x$ and $y = x^2 + 2x + 5$

4 Solve the simultaneous equations $x + y = 1$ and $x^2 + y^2 = 25$ graphically.

Extend

5 **a** Solve the simultaneous equations $2x + y = 3$ and $x^2 + y = 4$

i graphically

ii algebraically to 2 decimal places.

b Which method gives the more accurate solutions? Explain your answer.

Answers

1 **a** $x = 2, y = 5$
b $x = 2, y = -3$
c $x = -0.5, y = 2.5$

2 **a** $x = -2, y = 2$
b $x = 0.5, y = 0.5$
c $x = -1, y = -2$

3 **a** $x = 1, y = 0$ and $x = 4, y = 3$
b $x = -2, y = 7$ and $x = 2, y = -5$
c $x = -2, y = 5$ and $x = -1, y = 4$

4 $x = -3, y = 4$ and $x = 4, y = -3$

5 **a** **i** $x = 2.5, y = -2$ and $x = -0.5, y = 4$
ii $x = 2.41, y = -1.83$ and $x = -0.41, y = 3.83$
b Solving algebraically gives the more accurate solutions as the solutions from the graph are only estimates, based on the accuracy of your graph.

By the end of this task you will:

- have gained knowledge and understanding of sketching graphs
- be able to sketch cubic and reciprocal graphs

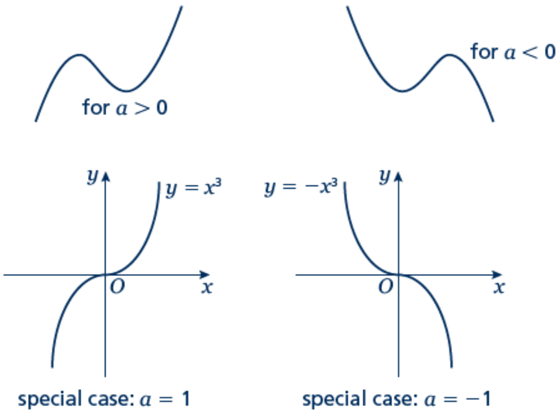
Task 3:

Sketching cubic and reciprocal graphs

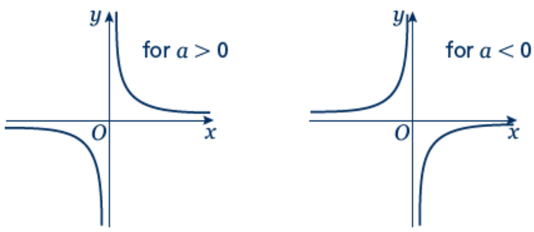
A LEVEL LINKS
Graphs – cubic, quartic and reciprocal

Key points

- The graph of a cubic function, which can be written in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has one of the shapes shown here.



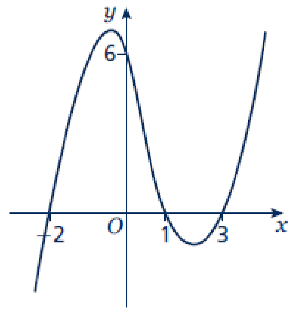
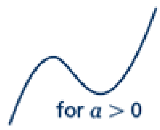
- The graph of a reciprocal function of the form $y = \frac{a}{x}$ has one of the shapes shown here.



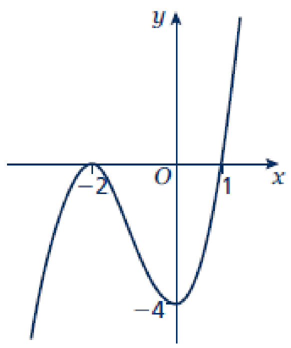
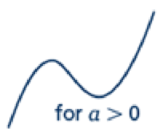
- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- Where appropriate, mark and label the asymptotes on the graph.
- Asymptotes are lines (usually horizontal or vertical) which the curve gets closer to but never touches or crosses. Asymptotes usually occur with reciprocal functions. For example, the asymptotes for the graph of $y = \frac{a}{x}$ are the two axes (the lines $y = 0$ and $x = 0$).
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- A double root is when two of the solutions are equal. For example $(x – 3)^2(x + 2)$ has a double root at $x = 3$.
- When there is a double root, this is one of the turning points of a cubic function.

Examples

Example 1 Sketch the graph of $y = (x - 3)(x - 1)(x + 2)$

To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.	
When $x = 0$, $y = (0 - 3)(0 - 1)(0 + 2)$ $= (-3) \times (-1) \times 2 = 6$ The graph intersects the y-axis at $(0, 6)$ When $y = 0$, $(x - 3)(x - 1)(x + 2) = 0$ So $x = 3$, $x = 1$ or $x = -2$ The graph intersects the x-axis at $(-2, 0)$, $(1, 0)$ and $(3, 0)$  	**1** Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Make sure you get the coordinates the right way around, (x, y). **2** Solve the equation by solving $x - 3 = 0$, $x - 1 = 0$ and $x + 2 = 0$ **3** Sketch the graph. $a = 1 > 0$ so the graph has the shape: 

Example 2 Sketch the graph of $y = (x + 2)^2(x - 1)$

To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.	
When $x = 0$, $y = (0 + 2)^2(0 - 1)$ $= 2^2 \times (-1) = -4$ The graph intersects the y-axis at $(0, -4)$ When $y = 0$, $(x + 2)^2(x - 1) = 0$ So $x = -2$ or $x = 1$ $(-2, 0)$ is a turning point as $x = -2$ is a double root. The graph crosses the x-axis at $(1, 0)$  	**1** Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. **2** Solve the equation by solving $x + 2 = 0$ and $x - 1 = 0$ **3** $a = 1 > 0$ so the graph has the shape: 

Practice

1 Here are six equations.

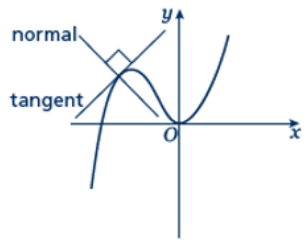
A $y = \dfrac{5}{x}$ **B** $y = x^2 + 3x - 10$ **C** $y = x^3 + 3x^2$

D $y = 1 - 3x^2 - x^3$ **E** $y = x^3 - 3x^2 - 1$ **F** $x + y = 5$

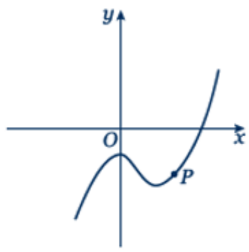
> **Hint**
>
> Find where each of the cubic equations cross the y-axis.

Here are six graphs.

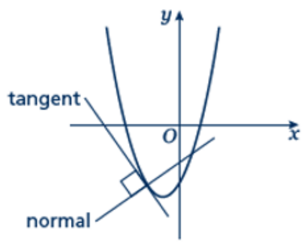
i



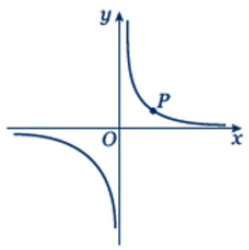
ii



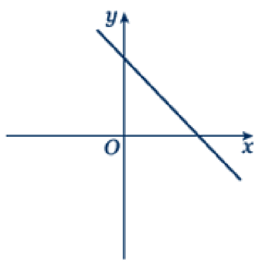
iii



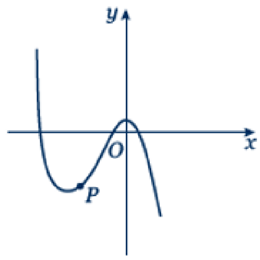
iv



v



vi



a Match each graph to its equation.

b Copy the graphs ii, iv and vi and draw the tangent and normal each at point P.

Sketch the following graphs

2 $y = 2x^3$

3 $y = x(x - 2)(x + 2)$

4 $y = (x + 1)(x + 4)(x - 3)$

5 $y = (x + 1)(x - 2)(1 - x)$

6 $y = (x - 3)^2(x + 1)$

7 $y = (x - 1)^2(x - 2)$

8 $y = \dfrac{3}{x}$

> **Hint:** Look at the shape of $y = \dfrac{a}{x}$ in the second key point.

9 $y = -\dfrac{2}{x}$

Extend

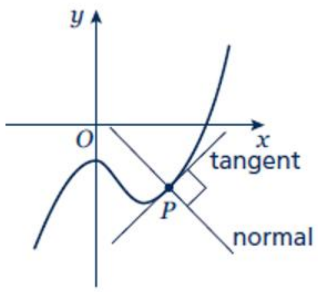
10 Sketch the graph of $y = \dfrac{1}{x+2}$

11 Sketch the graph of $y = \dfrac{1}{x-1}$

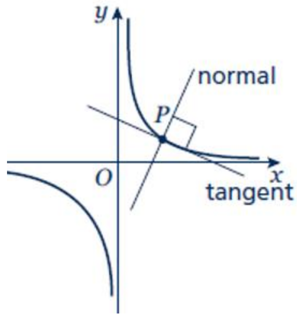
Answers

1 **a** i – C
ii – E
iii – B
iv – A
v – F
vi – D

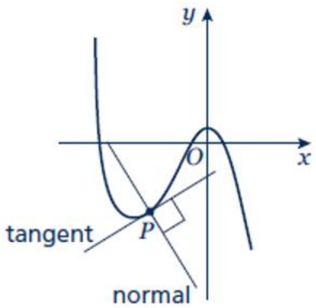
b **ii**



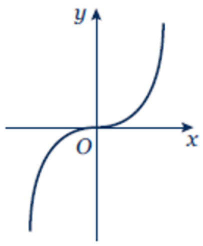
iv



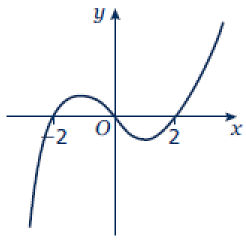
vi



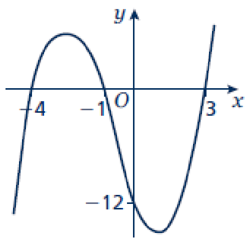
2



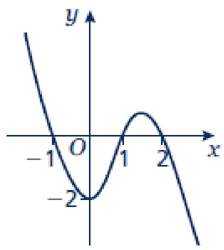
3



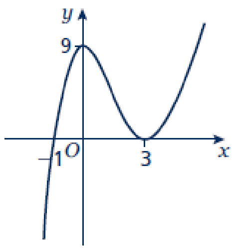
4



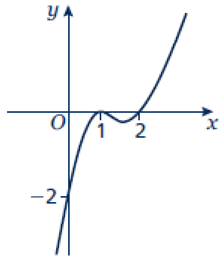
5



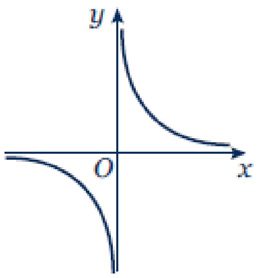
6



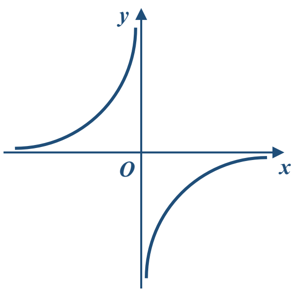
7



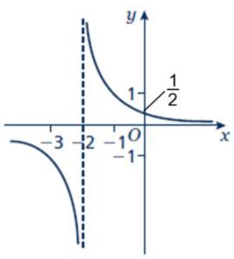
8



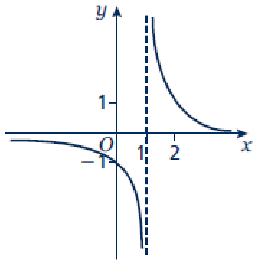
9



10



11



By the end of this task you will:

- have gained knowledge and understanding of graph transformations
- be able to translate graphs

Task 4:

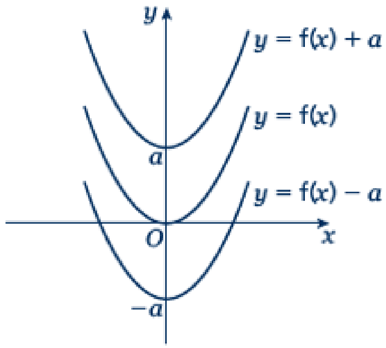
Translating graphs

A LEVEL LINKS
Transformations – transforming graphs – f(x) notation

Key points

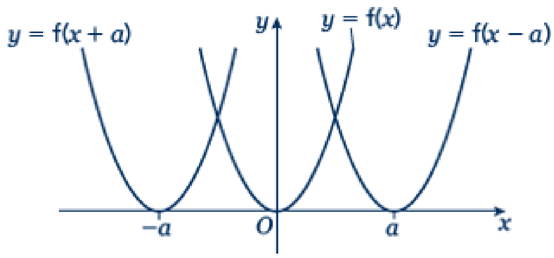
- The transformation $y = \mathrm{f}(x) \pm a$ is a translation of $y = \mathrm{f}(x)$ parallel to the y-axis; it is a vertical translation.

  As shown on the graph,
  - $y = \mathrm{f}(x) + a$ translates $y = \mathrm{f}(x)$ up
  - $y = \mathrm{f}(x) - a$ translates $y = \mathrm{f}(x)$ down.



- The transformation $y = \mathrm{f}(x \pm a)$ is a translation of $y = \mathrm{f}(x)$ parallel to the x-axis; it is a horizontal translation.

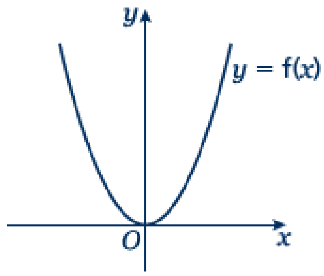
  As shown on the graph,
  - $y = \mathrm{f}(x + a)$ translates $y = \mathrm{f}(x)$ to the left
  - $y = \mathrm{f}(x - a)$ translates $y = \mathrm{f}(x)$ to the right.

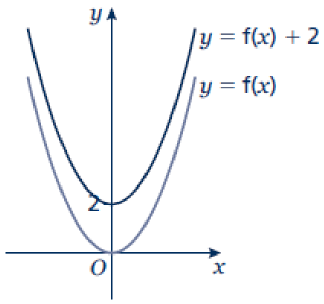


Examples

Example 1 The graph shows the function $y = \mathrm{f}(x)$.

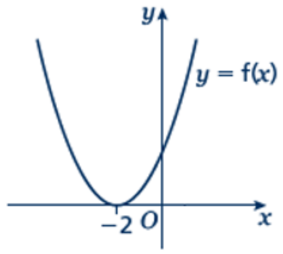
Sketch the graph of $y = \mathrm{f}(x) + 2$.

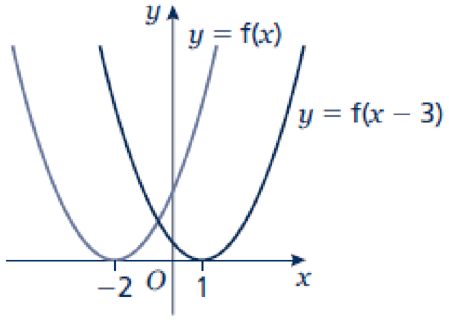


	For the function $y = \mathrm{f}(x) + 2$ translate the function $y = \mathrm{f}(x)$ 2 units up.

Example 2 The graph shows the function $y = \text{f}(x)$.

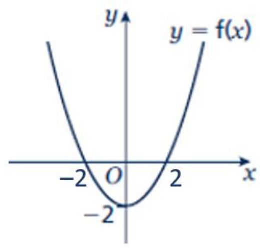
Sketch the graph of $y = \text{f}(x - 3)$.



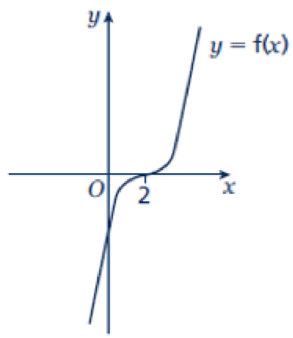
	For the function $y = \text{f}(x - 3)$ translate the function $y = \text{f}(x)$ 3 units right.

Practice

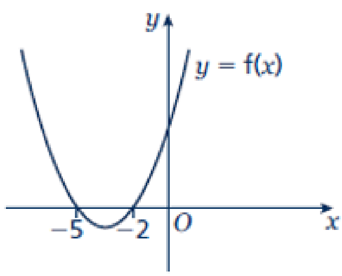
1 The graph shows the function $y = \text{f}(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = \text{f}(x) + 4$ and $y = \text{f}(x + 2)$.



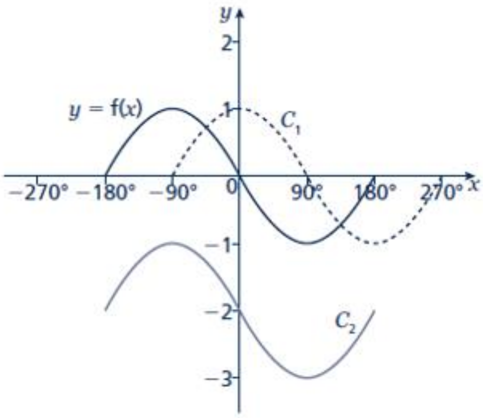
2 The graph shows the function $y = \text{f}(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = \text{f}(x + 3)$ and $y = \text{f}(x) - 3$.



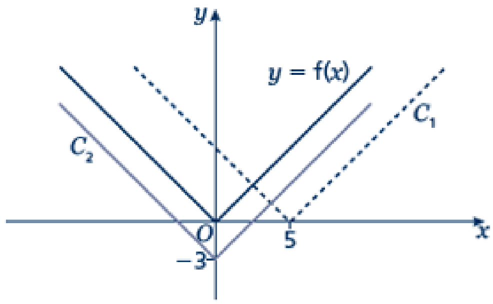
3 The graph shows the function $y = \text{f}(x)$.
Copy the graph and on the same axes sketch the graph of $y = \text{f}(x - 5)$.



4 The graph shows the function $y = f(x)$ and two transformations of $y = f(x)$, labelled C_1 and C_2. Write down the equations of the translated curves C_1 and C_2 in function form.



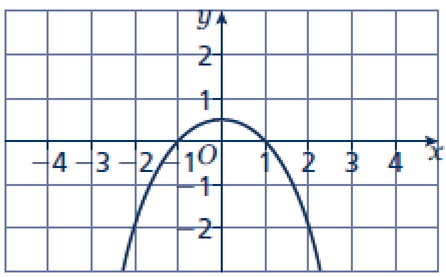
5 The graph shows the function $y = f(x)$ and two transformations of $y = f(x)$, labelled C_1 and C_2. Write down the equations of the translated curves C_1 and C_2 in function form.



6 The graph shows the function $y = f(x)$.

a Sketch the graph of $y = f(x) + 2$

b Sketch the graph of $y = f(x + 2)$



(Answers after task 5)

By the end of this task you will:

- have gained knowledge and understanding of graph transformations
- be able to stretch graphs

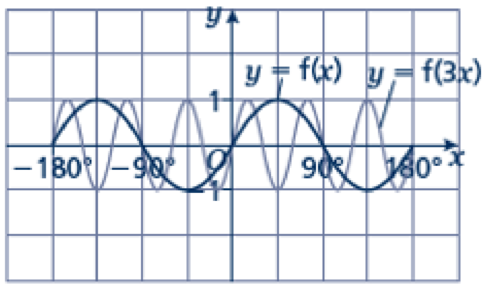
Task 5:

Stretching graphs

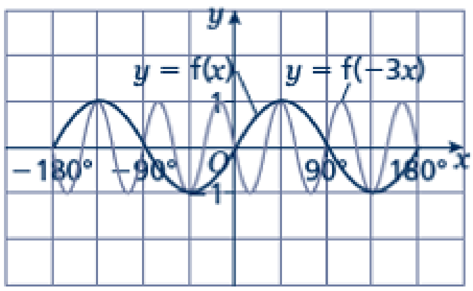
A LEVEL LINKS
Transformations – transforming graphs – f(x) notation

Key points

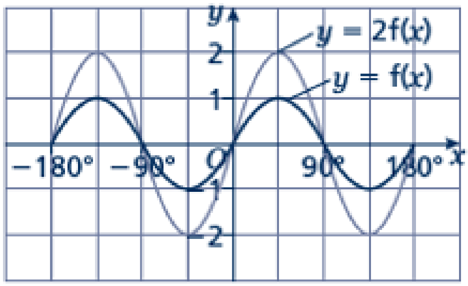
- The transformation $y = f(ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{a}$ parallel to the x-axis.



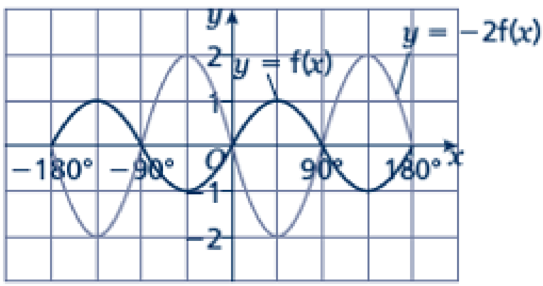
- The transformation $y = f(-ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{a}$ parallel to the x-axis and then a reflection in the y-axis.



- The transformation $y = af(x)$ is a vertical stretch of $y = f(x)$ with scale factor a parallel to the y-axis.



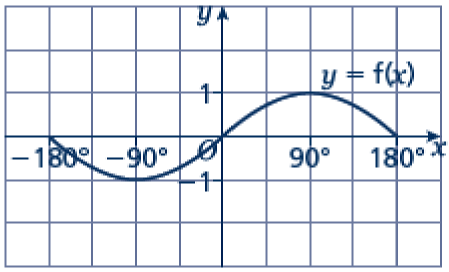
- The transformation $y = -af(x)$ is a vertical stretch of $y = f(x)$ with scale factor a parallel to the y-axis and then a reflection in the x-axis.

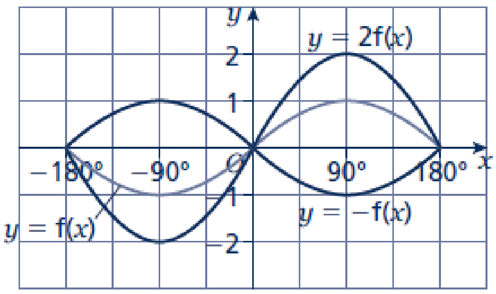


Examples

Example 3 The graph shows the function $y = f(x)$.

Sketch and label the graphs of $y = 2f(x)$ and $y = -f(x)$.



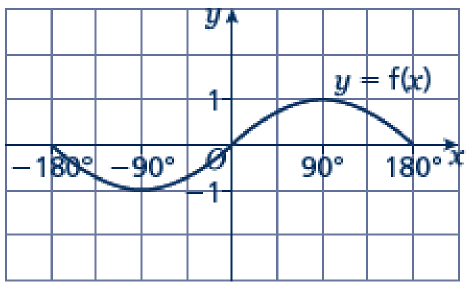


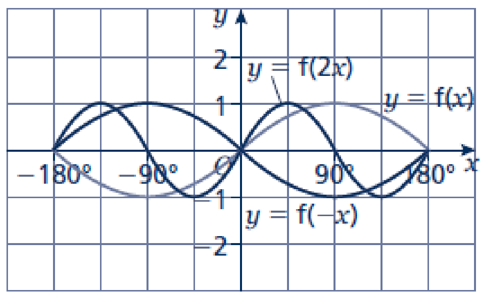
The function $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2 parallel to the y-axis.

The function $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis.

Example 4 The graph shows the function $y = f(x)$.

Sketch and label the graphs of $y = f(2x)$ and $y = f(-x)$.





The function $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$ parallel to the x-axis.

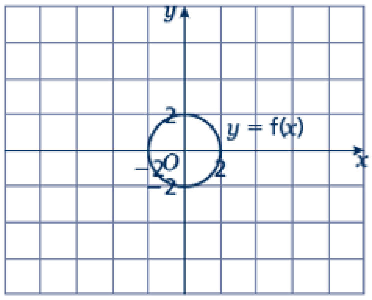
The function $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.

Practice

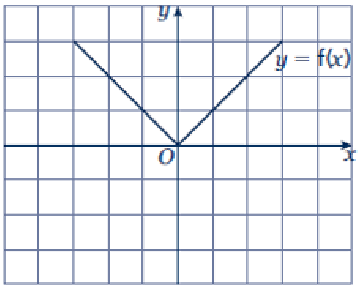
7 The graph shows the function $y = f(x)$.

a Copy the graph and on the same axes sketch and label the graph of $y = 3f(x)$.

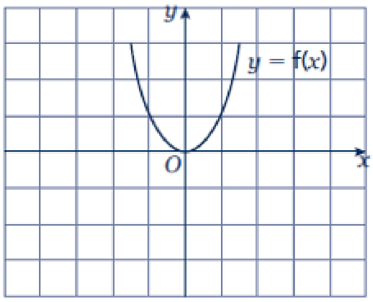
b Make another copy of the graph and on the same axes sketch and label the graph of $y = f(2x)$.



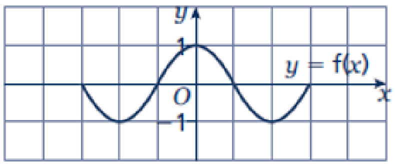
8 The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = -2f(x)$ and $y = f(3x)$.



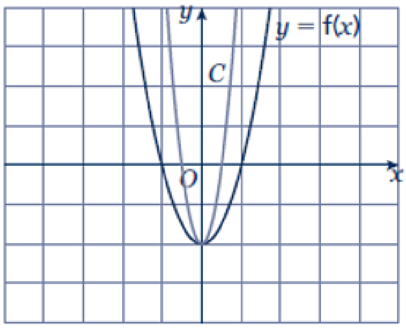
9 The graph shows the function $y = f(x)$.
Copy the graph and, on the same axes, sketch and label the graphs of
$y = -f(x)$ and $y = f\left(\frac{1}{2}x\right)$.



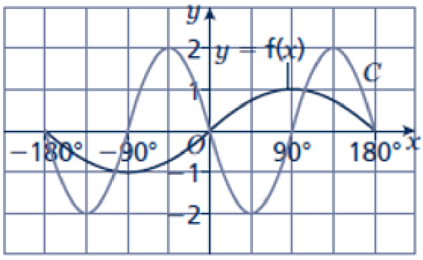
10 The graph shows the function $y = f(x)$.
Copy the graph and, on the same axes, sketch the graph of $y = -f(2x)$.



11 The graph shows the function $y = f(x)$ and a transformation, labelled C.
Write down the equation of the translated curve C in function form.



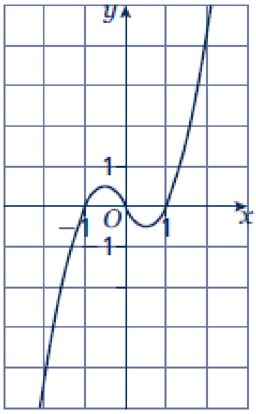
12 The graph shows the function $y = \text{f}(x)$ and a transformation labelled C.
Write down the equation of the translated curve C in function form.



13 The graph shows the function $y = \text{f}(x)$.

a Sketch the graph of $y = -\text{f}(x)$.

b Sketch the graph of $y = 2\text{f}(x)$.



Extend

14 **a** Sketch and label the graph of $y = \text{f}(x)$, where $\text{f}(x) = (x - 1)(x + 1)$.

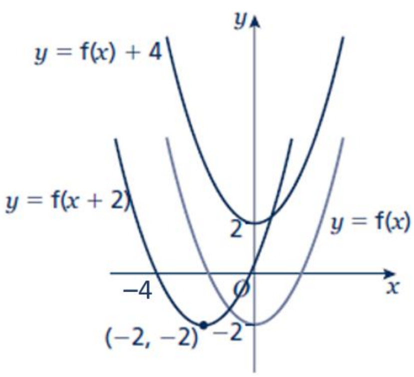
b On the same axes, sketch and label the graphs of $y = \text{f}(x) - 2$ and $y = \text{f}(x + 2)$.

15 **a** Sketch and label the graph of $y = \text{f}(x)$, where $\text{f}(x) = -(x + 1)(x - 2)$.

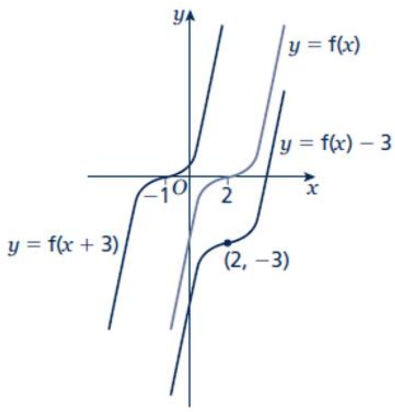
b On the same axes, sketch and label the graph of $y = \text{f}\left(-\frac{1}{2}x\right)$.

Answers

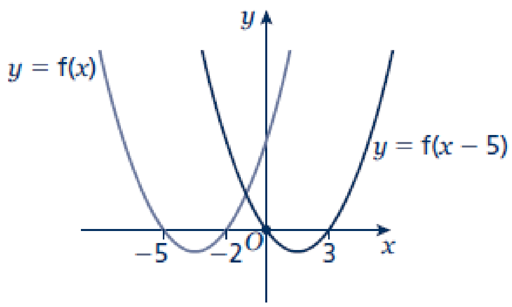
1



2



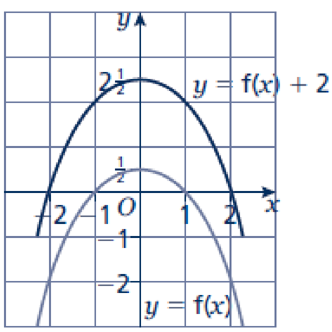
3



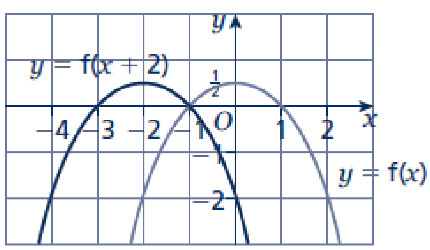
4 C_1: $y = f(x - 90°)$
C_2: $y = f(x) - 2$

5 C_1: $y = f(x - 5)$
C_2: $y = f(x) - 3$

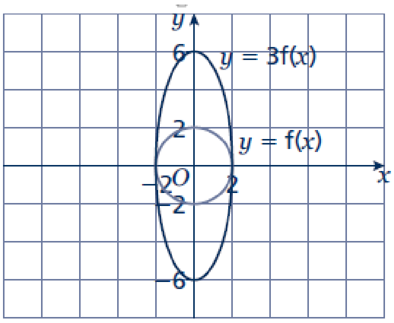
6 **a**



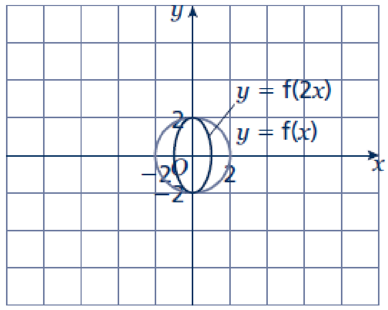
b



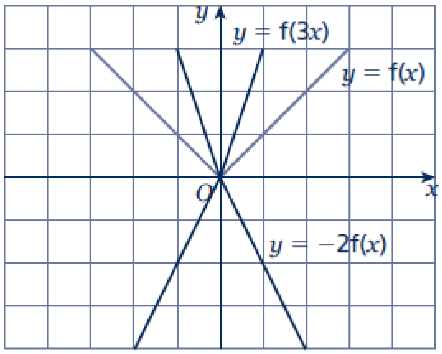
7 **a**



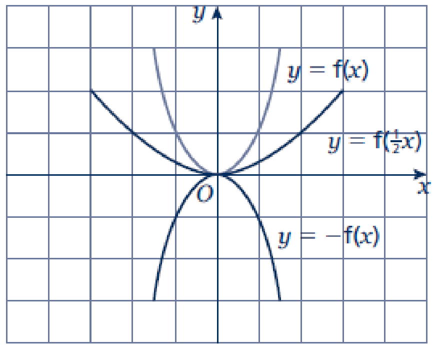
b



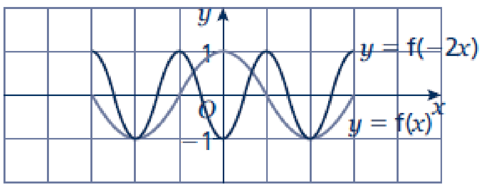
8



9



10

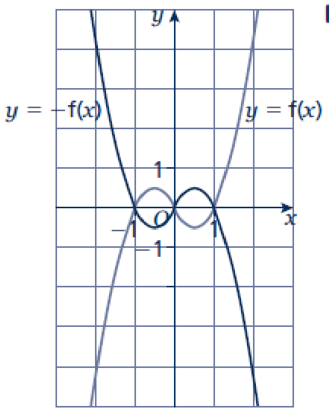


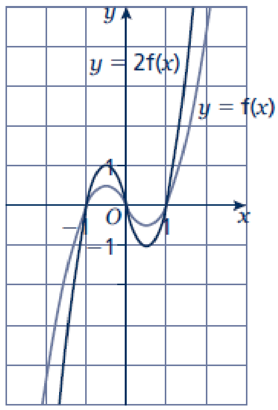
11 $y = \mathrm{f}(2x)$

12 $y = -2\mathrm{f}(2x)$ or $y = 2\mathrm{f}(-2x)$

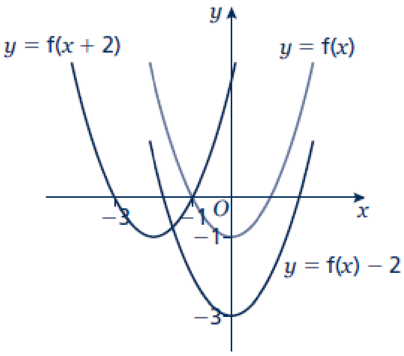
13 **a**

b

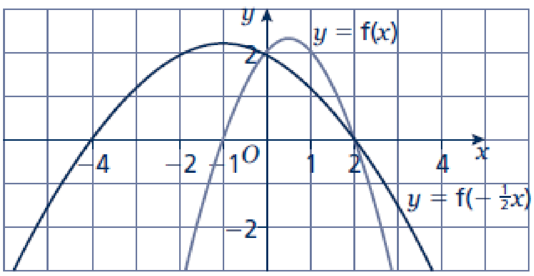

y
y = −f(x)
y = f(x)
1
O
−1
x


y
y = 2f(x)
y = f(x)
1
O
−1
x

14


y
y = f(x + 2)
y = f(x)
−3
−1
O
x
−1
y = f(x) − 2
−3

15


y
y = f(x)
2
−4
−2
−1
O
1
2
4
x
y = f(−½x)
−2

By the end of this task you will:

- ❑ have gained knowledge and understanding of straight-line graphs
- ❑ be able to find the equations of a line
- ❑ be able to find the gradient and y-intercept of a

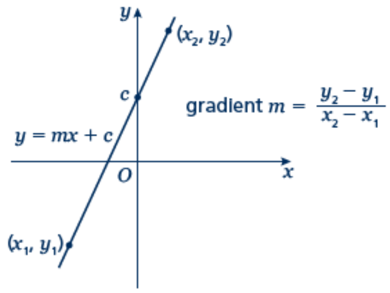
Task 6:

Straight line graphs

A LEVEL LINKS
Straight-line graphs, parallel/perpendicular, length and area problems

Key points

- A straight line has the equation $y = mx + c$, where m is the gradient and c is the y-intercept (where $x = 0$).
- The equation of a straight line can be written in the form $ax + by + c = 0$, where a, b and c are integers.
- When given the coordinates (x_1, y_1) and (x_2, y_2) of two points on a line the gradient is calculated using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$



Examples

Example 1 A straight line has gradient $-\frac{1}{2}$ and y-intercept 3.

Write the equation of the line in the form $ax + by + c = 0$.

$m = -\frac{1}{2}$ and $c = 3$ So $y = -\frac{1}{2}x + 3$ $\frac{1}{2}x + y - 3 = 0$ $x + 2y - 6 = 0$	**1** A straight line has equation $y = mx + c$. Substitute the gradient and y-intercept given in the question into this equation. **2** Rearrange the equation so all the terms are on one side and 0 is on the other side. **3** Multiply both sides by 2 to eliminate the denominator.

Example 2 Find the gradient and the y-intercept of the line with the equation $3y - 2x + 4 = 0$.

$3y - 2x + 4 = 0$ $3y = 2x - 4$ $y = \frac{2}{3}x - \frac{4}{3}$ Gradient $= m = \frac{2}{3}$ y-intercept $= c = -\frac{4}{3}$	**1** Make y the subject of the equation. **2** Divide all the terms by three to get the equation in the form $y = \ldots$ **3** In the form $y = mx + c$, the gradient is m and the y-intercept is c.

Example 3 Find the equation of the line which passes through the point (5, 13) and has gradient 3.

$m = 3$ $y = 3x + c$ $13 = 3 \times 5 + c$ $13 = 15 + c$ $c = -2$ $y = 3x - 2$	**1** Substitute the gradient given in the question into the equation of a straight-line $y = mx + c$. **2** Substitute the coordinates $x = 5$ and $y = 13$ into the equation. **3** Simplify and solve the equation. **4** Substitute $c = -2$ into the equation $y = 3x + c$

Example 4 Find the equation of the line passing through the points with coordinates (2, 4) and (8, 7).

$x_1 = 2,\ x_2 = 8,\ y_1 = 4$ and $y_2 = 7$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{8-2} = \frac{3}{6} = \frac{1}{2}$ $y = \frac{1}{2}x + c$ $4 = \frac{1}{2} \times 2 + c$ $c = 3$ $y = \frac{1}{2}x + 3$	**1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line. **2** Substitute the gradient into the equation of a straight-line $y = mx + c$. **3** Substitute the coordinates of either point into the equation. **4** Simplify and solve the equation. **5** Substitute $c = 3$ into the equation $y = \frac{1}{2}x + c$

Practice

1 Find the gradient and the y-intercept of the following equations.

a $y = 3x + 5$ **b** $y = -\frac{1}{2}x - 7$

c $2y = 4x - 3$ **d** $x + y = 5$

e $2x - 3y - 7 = 0$ **f** $5x + y - 4 = 0$

> **Hint**
> Rearrange the equations to the form $y = mx + c$

2 Copy and complete the table, giving the equation of the line in the form $y = mx + c$.

Gradient	y-intercept	Equation of the line
5	0	
–3	2	

4	–7	

3 Find, in the form $ax + by + c = 0$ where a, b and c are integers, an equation for each of the lines with the following gradients and y-intercepts.

a gradient $-\frac{1}{2}$, y-intercept –7 **b** gradient 2, y-intercept 0

c gradient $\frac{2}{3}$, y-intercept 4 **d** gradient –1.2, y-intercept –2

4 Write an equation for the line which passes though the point (2, 5) and has gradient 4.

5 Write an equation for the line which passes through the point (6, 3) and has gradient $-\frac{2}{3}$

6 Write an equation for the line passing through each of the following pairs of points.

a (4, 5), (10, 17) **b** (0, 6), (–4, 8)

c (–1, –7), (5, 23) **d** (3, 10), (4, 7)

Extend

7 The equation of a line is $2y + 3x - 6 = 0$.
Write as much information as possible about this line.

Answers

1 **a** $m = 3, c = 5$ **b** $m = -\frac{1}{2}, c = -7$

c $m = 2, c = -\frac{3}{2}$ **d** $m = -1, c = 5$

e $m = \frac{2}{3}, c = -\frac{7}{3}$ or $-2\frac{1}{3}$ **f** $m = -5, c = 4$

2

Gradient	*y*-intercept	Equation of the line
5	0	$y = 5x$
–3	2	$y = -3x + 2$
4	–7	$y = 4x - 7$

3 **a** $x + 2y + 14 = 0$ **b** $2x - y = 0$

c $2x - 3y + 12 = 0$ **d** $6x + 5y + 10 = 0$

4 $y = 4x - 3$

5 $y = -\frac{2}{3}x + 7$

6 **a** $y = 2x - 3$ **b** $y = -\frac{1}{2}x + 6$

c $y = 5x - 2$ **d** $y = -3x + 19$

7 $y = -\frac{3}{2}x + 3$, the gradient is $-\frac{3}{2}$ and the *y*-intercept is 3.
The line intercepts the axes at (0, 3) and (2, 0).
Students may sketch the line or give coordinates that lie on the line such as $\left(1, \frac{3}{2}\right)$ or $(4, -3)$.

By the end of this task you will:

- have gained knowledge and understanding of parallel and perpendicular lines
- be able to find the equations of parallel and perpendicular lines
- be able to find the gradient and y-intercept of a parallel and perpendicular line

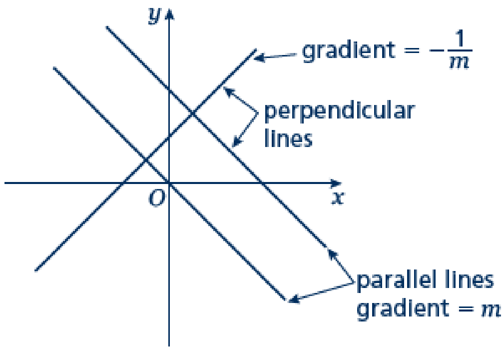
Task 7:

Parallel and perpendicular lines

A LEVEL LINKS
Straight-line graphs, parallel/perpendicular, length and area problems

Key points

- When lines are parallel they have the same gradient.
- A line perpendicular to the line with equation $y = mx + c$ has gradient $-\frac{1}{m}$.



Examples

Example 1 Find the equation of the line parallel to $y = 2x + 4$ which passes through the point (4, 9).

$y = 2x + 4$ $m = 2$ $y = 2x + c$	**1** As the lines are parallel they have the same gradient. **2** Substitute $m = 2$ into the equation of a straight-line $y = mx + c$.
$9 = 2 \times 4 + c$	**3** Substitute the coordinates into the equation $y = 2x + c$
$9 = 8 + c$ $c = 1$	**4** Simplify and solve the equation.
$y = 2x + 1$	**5** Substitute $c = 1$ into the equation $y = 2x + c$

Example 2 Find the equation of the line perpendicular to $y = 2x - 3$ which passes through the point (−2, 5).

$y = 2x - 3$ $m = 2$ $-\frac{1}{m} = -\frac{1}{2}$	**1** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.
$y = -\frac{1}{2}x + c$	**2** Substitute $m = -\frac{1}{2}$ into $y = mx + c$.
$5 = -\frac{1}{2} \times (-2) + c$	**3** Substitute the coordinates (–2, 5) into the equation $y = -\frac{1}{2}x + c$
$5 = 1 + c$ $c = 4$	**4** Simplify and solve the equation.
$y = -\frac{1}{2}x + 4$	**5** Substitute $c = 4$ into $y = -\frac{1}{2}x + c$.

Example 3 A line passes through the points (0, 5) and (9, −1).
Find the equation of the line which is perpendicular to the line and passes through its midpoint.

$x_1 = 0$, $x_2 = 9$, $y_1 = 5$ and $y_2 = -1$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{9-0}$ $= \frac{-6}{9} = -\frac{2}{3}$	**1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line.
$-\frac{1}{m} = \frac{3}{2}$	**2** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.
$y = \frac{3}{2}x + c$	**3** Substitute the gradient into the equation $y = mx + c$.
Midpoint = $\left(\frac{0+9}{2}, \frac{5+(-1)}{2}\right) = \left(\frac{9}{2}, 2\right)$	**4** Work out the coordinates of the midpoint of the line.
$2 = \frac{3}{2} \times \frac{9}{2} + c$	**5** Substitute the coordinates of the midpoint into the equation.
$c = -\frac{19}{4}$	**6** Simplify and solve the equation.
$y = \frac{3}{2}x - \frac{19}{4}$	**7** Substitute $c = -\frac{19}{4}$ into the equation $y = \frac{3}{2}x + c$.

Practice

1 Find the equation of the line parallel to each of the given lines and which passes through each of the given points.

a $y = 3x + 1$ (3, 2)

b $y = 3 - 2x$ (1, 3)

c $2x + 4y + 3 = 0$ (6, –3)

d $2y - 3x + 2 = 0$ (8, 20)

2 Find the equation of the line perpendicular to $y = \frac{1}{2}x - 3$ which passes through the point (–5, 3).

> **Hint**
> If $m = \frac{a}{b}$ then the negative reciprocal $-\frac{1}{m} = -\frac{b}{a}$

3 Find the equation of the line perpendicular to each of the given lines and which passes through each of the given points.

a $y = 2x - 6$ (4, 0)

b $y = -\frac{1}{3}x + \frac{1}{2}$ (2, 13)

c $x - 4y - 4 = 0$ (5, 15)

d $5y + 2x - 5 = 0$ (6, 7)

4 In each case find an equation for the line passing through the origin which is also perpendicular to the line joining the two points given.

a (4, 3), (–2, –9)

b (0, 3), (–10, 8)

Extend

5 Work out whether these pairs of lines are parallel, perpendicular or neither.

a $y = 2x + 3$
$y = 2x - 7$

b $y = 3x$
$2x + y - 3 = 0$

c $y = 4x - 3$
$4y + x = 2$

d $3x - y + 5 = 0$
$x + 3y = 1$

e $2x + 5y - 1 = 0$
$y = 2x + 7$

f $2x - y = 6$
$6x - 3y + 3 = 0$

6 The straight line $\mathbf{L_1}$ passes through the points A and B with coordinates (–4, 4) and (2, 1), respectively.

a Find the equation of $\mathbf{L_1}$ in the form $ax + by + c = 0$

The line $\mathbf{L_2}$ is parallel to the line $\mathbf{L_1}$ and passes through the point C with coordinates (–8, 3).

b Find the equation of $\mathbf{L_2}$ in the form $ax + by + c = 0$

The line $\mathbf{L_3}$ is perpendicular to the line $\mathbf{L_1}$ and passes through the origin.

c Find an equation of $\mathbf{L_3}$

Answers

1	**a**	$y = 3x - 7$	**b**	$y = -2x + 5$		
	c	$y = -\frac{1}{2}x$	**d**	$y = \frac{3}{2}x + 8$		
2	$y = -2x - 7$					
3	**a**	$y = -\frac{1}{2}x + 2$	**b**	$y = 3x + 7$		
	c	$y = -4x + 35$	**d**	$y = \frac{5}{2}x - 8$		
4	**a**	$y = -\frac{1}{2}x$	**b**	$y = 2x$		
5	**a**	Parallel	**b**	Neither	**c**	Perpendicular
	d	Perpendicular	**e**	Neither	**f**	Parallel
6	**a**	$x + 2y - 4 = 0$	**b**	$x + 2y + 2 = 0$	**c**	$y = 2x$

By the end of this task you will:

- have gained knowledge and understanding of Pythagoras theorem
- be able to find the missing length of a right-angled triangle

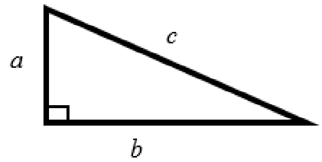
Task 8:

Pythagoras' theorem

A LEVEL LINKS
Straight-line graphs, parallel/perpendicular, length and area problems

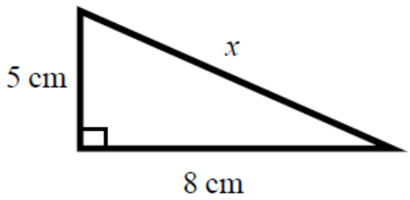
Key points

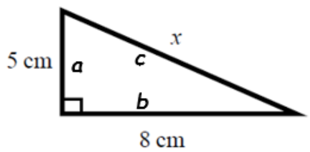
- In a right-angled triangle the longest side is called the hypotenuse.
- Pythagoras' theorem states that for a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.
  $c^2 = a^2 + b^2$



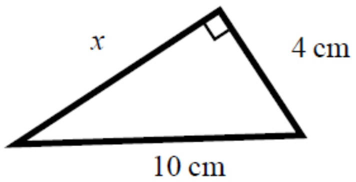
Examples

Example 1 Calculate the length of the hypotenuse.
Give your answer to 3 significant figures.



$c^2 = a^2 + b^2$ 	**1** Always start by stating the formula for Pythagoras' theorem and labelling the hypotenuse c and the other two sides a and b.
$x^2 = 5^2 + 8^2$ $x^2 = 25 + 64$ $x^2 = 89$ $x = \sqrt{89}$	**2** Substitute the values of a, b and c into the formula for Pythagoras' theorem. **3** Use a calculator to find the square root.
$x = 9.433\,981\,13...$ $x = 9.43$ cm	**4** Round your answer to 3 significant figures and write the units with your answer.

Example 2 Calculate the length x.
Give your answer in surd form.

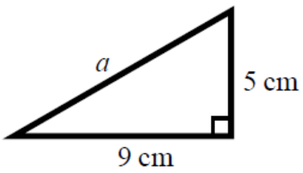


$c^2 = a^2 + b^2$ $10^2 = x^2 + 4^2$ $100 = x^2 + 16$ $x^2 = 84$ $x = \sqrt{84}$ $x = 2\sqrt{21}$ cm	**1** Always start by stating the formula for Pythagoras' theorem. **2** Substitute the values of a, b and c into the formula for Pythagoras' theorem. **3** Simplify the surd where possible and write the units in your answer.

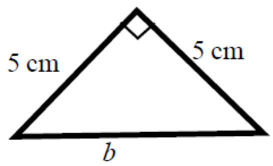
Practice

1 Work out the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

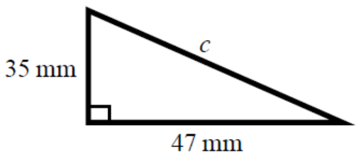
a



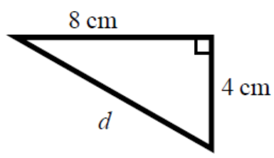
b



c

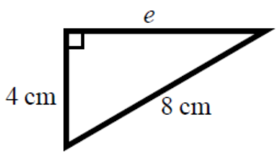


d

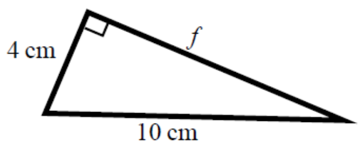


2 Work out the length of the unknown side in each triangle.
Give your answers in surd form.

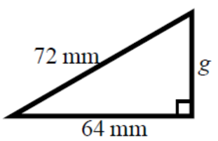
a



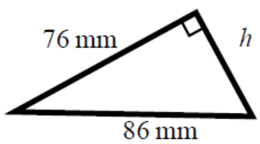
b



c

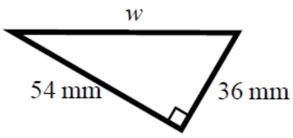


d

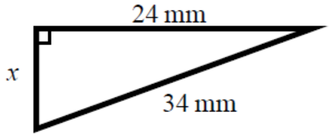


3 Work out the length of the unknown side in each triangle.
Give your answers in surd form.

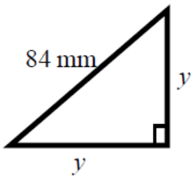
a



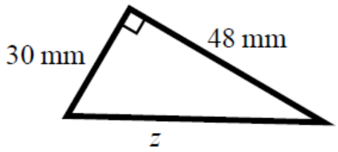
b



c



d



4 A rectangle has length 84 mm and width 45 mm.
Calculate the length of the diagonal of the rectangle.
Give your answer correct to 3 significant figures.

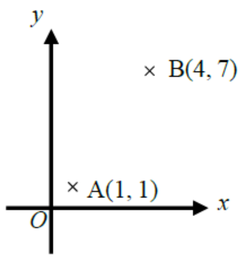
> **Hint**
>
> Draw a sketch of the rectangle.

Extend

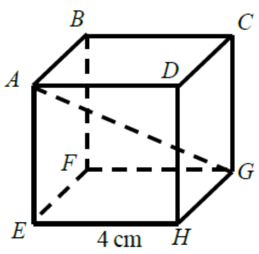
5 A yacht is 40 km due North of a lighthouse.
A rescue boat is 50 km due East of the same lighthouse.
Work out the distance between the yacht and the rescue boat.
Give your answer correct to 3 significant figures.

> **Hint**
>
> Draw a diagram using the information given in the question.

6 Points A and B are shown on the diagram.
Work out the length of the line AB.
Give your answer in surd form.



7 A cube has length 4 cm.
Work out the length of the diagonal AG.
Give your answer in surd form.



Answers

1 **a** 10.3 cm **b** 7.07 cm

c 58.6 mm **d** 8.94 cm

2 **a** $4\sqrt{3}$ cm **b** $2\sqrt{21}$ cm

c $8\sqrt{17}$ mm **d** $18\sqrt{5}$ mm

3 **a** $18\sqrt{13}$ mm **b** $2\sqrt{145}$ mm

c $42\sqrt{2}$ mm **d** $6\sqrt{89}$ mm

4 95.3 mm

5 64.0 km

6 $3\sqrt{5}$ units

7 $4\sqrt{3}$ cm

By the end of this task you will:

- have gained knowledge and understanding of circle theorems
- be able to use circle theorems in problem solving

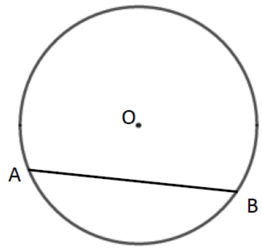
Task 9:

Circle theorems

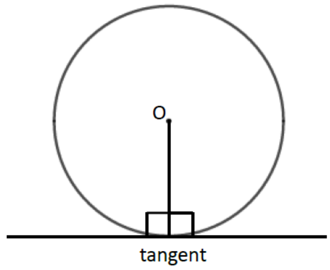
A LEVEL LINKS
Circles – equation of a circle, geometric problems on a grid

Key points

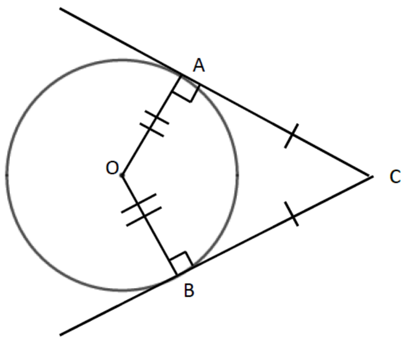
- A chord is a straight line joining two points on the circumference of a circle.
  So AB is a chord.



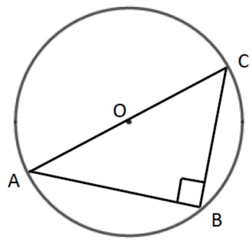
- A tangent is a straight line that touches the circumference of a circle at only one point.
  The angle between a tangent and the radius is 90°.



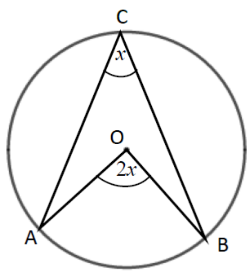
- Two tangents on a circle that meet at a point outside the circle are equal in length.
  So AC = BC.



- The angle in a semicircle is a right angle.
  So angle ABC = 90°.

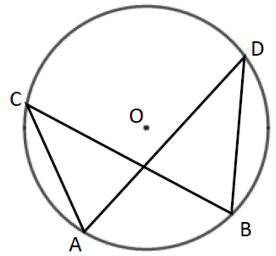


- When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.

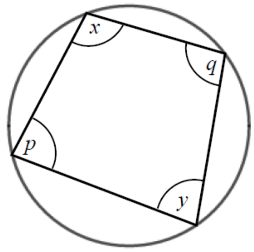


So angle AOB = 2 × angle ACB.

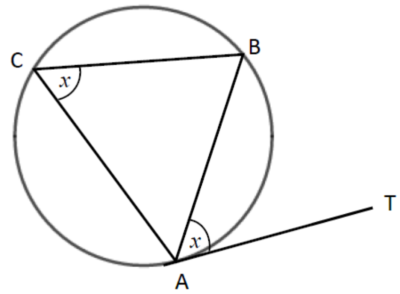
- Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.
  So angle ACB = angle ADB and
  angle CAD = angle CBD.



- A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle.
  Opposite angles in a cyclic quadrilateral total 180°.
  So $x + y = 180°$ and $p + q = 180°$.

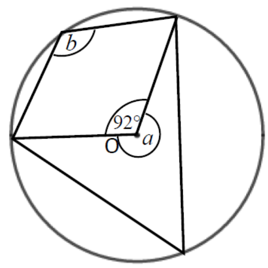


- The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem.
  So angle BAT = angle ACB.



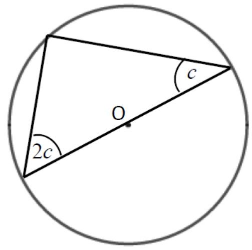
Examples

Example 1 Work out the size of each angle marked with a letter.
Give reasons for your answers.



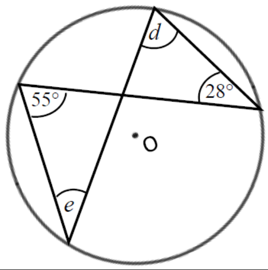
Angle $a = 360° - 92°$ $= 268°$ as the angles in a full turn total 360°. Angle $b = 268° \div 2$ $= 134°$ as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.	**1** The angles in a full turn total 360°. **2** Angles a and b are subtended by the same arc, so angle b is half of angle a.

Example 2 Work out the size of the angles in the triangle.
Give reasons for your answers.



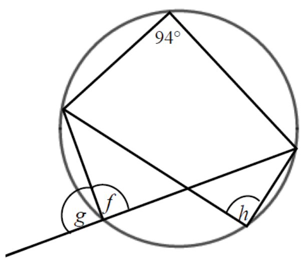
Angles are 90°, 2c and c. $90° + 2c + c = 180°$ $90° + 3c = 180°$ $3c = 90°$ $c = 30°$ $2c = 60°$ The angles are 30°, 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180°.	**1** The angle in a semicircle is a right angle. **2** Angles in a triangle total 180°. **3** Simplify and solve the equation.

Example 3 Work out the size of each angle marked with a letter.
Give reasons for your answers.



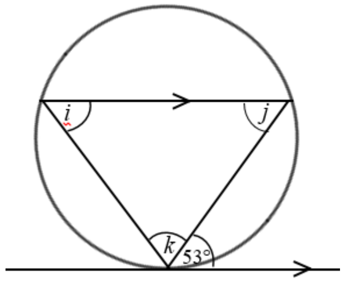
Angle d = 55° as angles subtended by the same arc are equal. Angle e = 28° as angles subtended by the same arc are equal.	**1** Angles subtended by the same arc are equal so angle 55° and angle d are equal. **2** Angles subtended by the same arc are equal so angle 28° and angle e are equal.

Example 4 Work out the size of each angle marked with a letter.
Give reasons for your answers.



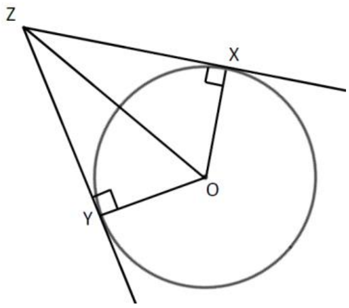
Angle $f = 180° - 94°$ $= 86°$ as opposite angles in a cyclic quadrilateral total 180°.	**1** Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle f total 180°. *(continued on next page)*
Angle $g = 180° - 86°$ $= 84°$ as angles on a straight line total 180°. Angle h = angle $f = 86°$ as angles subtended by the same arc are equal.	**2** Angles on a straight line total 180° so angle f and angle g total 180°. **3** Angles subtended by the same arc are equal so angle f and angle h are equal.

Example 5 Work out the size of each angle marked with a letter. Give reasons for your answers.



Angle $i = 53°$ because of the alternate segment theorem. Angle $j = 53°$ because it is the alternate angle to 53°. Angle $k = 180° - 53° - 53°$ $= 74°$ as angles in a triangle total 180°.	**1** The angle between a tangent and chord is equal to the angle in the alternate segment. **2** As there are two parallel lines, angle 53° is equal to angle j because they are alternate angles. **3** The angles in a triangle total 180°, so $i + j + k = 180°$.

Example 6 XZ and YZ are two tangents to a circle with centre O. Prove that triangles XZO and YZO are congruent.

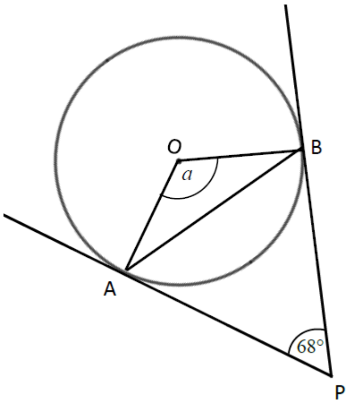


Angle OXZ = 90° and angle OYZ = 90° as the angles in a semicircle are right angles. OZ is a common line and is the hypotenuse in both triangles. OX = OY as they are radii of the same circle. So triangles XZO and YZO are congruent, RHS.	For two triangles to be congruent you need to show one of the following. • All three corresponding sides are equal (SSS). • Two corresponding sides and the included angle are equal (SAS). • One side and two corresponding angles are equal (ASA). • A right angle, hypotenuse and a shorter side are equal (RHS).

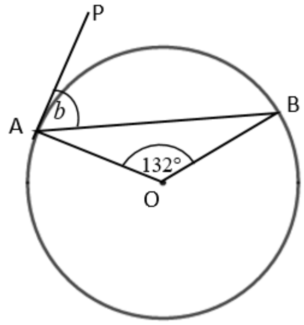
Practice

1 Work out the size of each angle marked with a letter.
Give reasons for your answers.

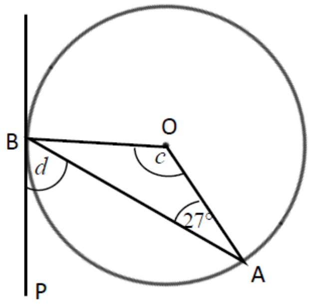
a



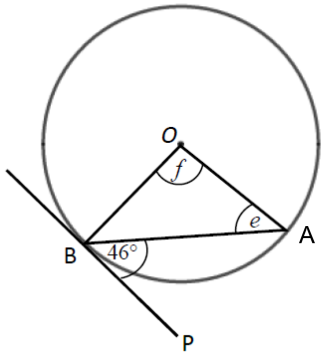
b



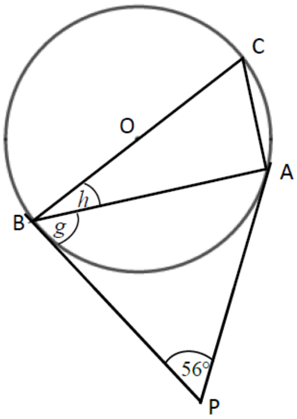
c



d

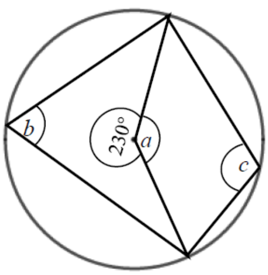


e

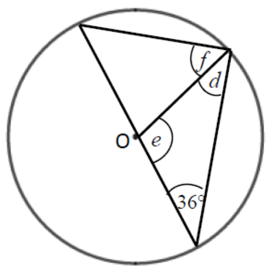


2 Work out the size of each angle marked with a letter.
Give reasons for your answers.

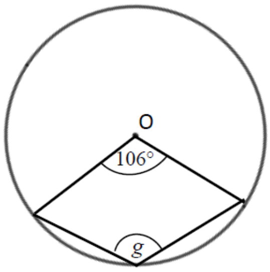
a



b



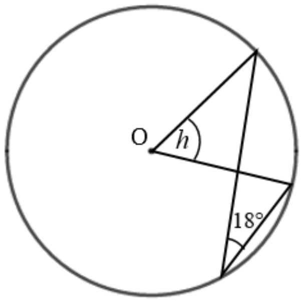
c



Hint

The reflex angle at point O and angle g are subtended by the same arc. So the reflex angle is twice the size of angle g.

d

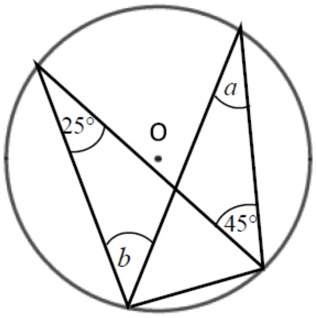


Hint

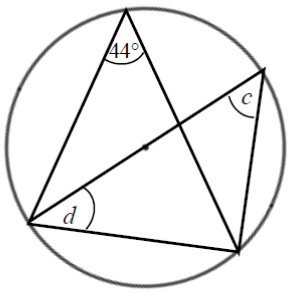
Angle 18° and angle h are subtended by the same arc.

3 Work out the size of each angle marked with a letter.
Give reasons for your answers.

a



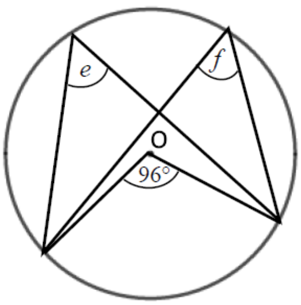
b



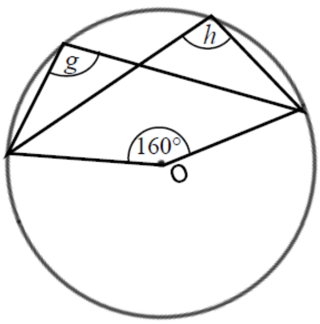
Hint

One of the angles is in a semicircle.

c

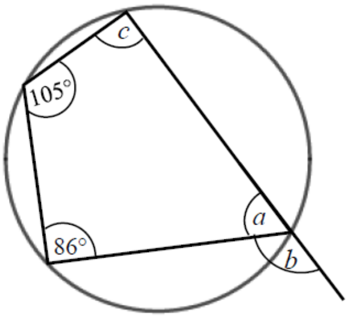


d



4 Work out the size of each angle marked with a letter.
Give reasons for your answers.

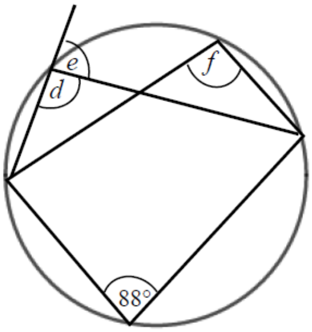
a



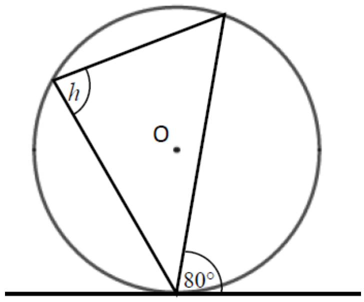
Hint

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

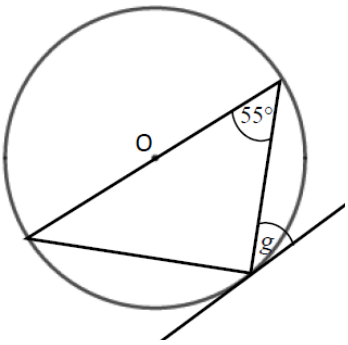
b



c



d



Hint

One of the angles is in a semicircle.

Extend

5 Prove the alternate segment theorem.

Answers

1 **a** $a = 112°$, angle OAP = angle OBP = 90° and angles in a quadrilateral total 360°.

b $b = 66°$, triangle OAB is isosceles, Angle OAP = 90° as AP is tangent to the circle.

c $c = 126°$, triangle OAB is isosceles.
$d = 63°$, Angle OBP = 90° as BP is tangent to the circle.

d $e = 44°$, the triangle is isosceles, so angles *e* and angle OBA are equal. The angle OBP = 90° as BP is tangent to the circle.
$f = 92°$, the triangle is isosceles.

e $g = 62°$, triangle ABP is isosceles as AP and BP are both tangents to the circle.
$h = 28°$, the angle OBP = 90°.

2 **a** $a = 130°$, angles in a full turn total 360°.
$b = 65°$, the angle at the centre of a circle is twice the angle at the circumference.
$c = 115°$, opposite angles in a cyclic quadrilateral total 180°.

b $d = 36°$, isosceles triangle.
$e = 108°$, angles in a triangle total 180°.
$f = 54°$, angle in a semicircle is 90°.

c $g = 127°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.

d $h = 36°$, the angle at the centre of a circle is twice the angle at the circumference.

3 **a** $a = 25°$, angles in the same segment are equal.
$b = 45°$, angles in the same segment are equal.

b $c = 44°$, angles in the same segment are equal.
$d = 46°$, the angle in a semicircle is 90° and the angles in a triangle total 180°.

c $e = 48°$, the angle at the centre of a circle is twice the angle at the circumference.
$f = 48°$, angles in the same segment are equal.

d $g = 100°$, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.
$h = 100°$, angles in the same segment are equal.

4 **a** $a = 75°$, opposite angles in a cyclic quadrilateral total 180°.
$b = 105°$, angles on a straight line total 180°.
$c = 94°$, opposite angles in a cyclic quadrilateral total 180°.

b $d = 92°$, opposite angles in a cyclic quadrilateral total 180°.
$e = 88°$, angles on a straight line total 180°.
$f = 92°$, angles in the same segment are equal.

c $h = 80°$, alternate segment theorem.

d $g = 35°$, alternate segment theorem and the angle in a semicircle is 90°.

5 Angle BAT = x.

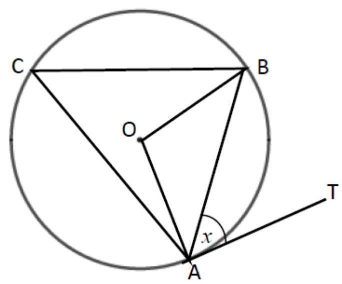
Angle OAB = 90° − x because the angle between the tangent and the radius is 90°.

OA = OB because radii are equal.

Angle OAB = angle OBA because the base of isosceles triangles are equal.

Angle AOB = 180° − (90° − x) − (90° − x) = 2x because angles in a triangle total 180°.

Angle ACB = 2x ÷ 2 = x because the angle at the centre is twice the angle at the circumference.



By the end of this task you will:

- ❑ have gained knowledge and understanding of trigonometry
- ❑ be able to find a missing side
- ❑ be able to find a missing angle

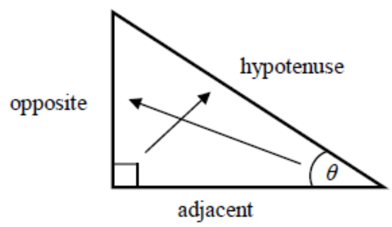
Task 10:

Trigonometry in right-angled triangles

A LEVEL LINKS
Trigonometric ratios and graphs

Key points

- In a right-angled triangle:
    - the side opposite the right angle is called the hypotenuse
    - the side opposite the angle θ is called the opposite
    - the side next to the angle θ is called the adjacent.



- In a right-angled triangle:
    - the ratio of the opposite side to the hypotenuse is the sine of angle θ, $\sin\theta = \frac{\text{opp}}{\text{hyp}}$
    - the ratio of the adjacent side to the hypotenuse is the cosine of angle θ, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$
    - the ratio of the opposite side to the adjacent side is the tangent of angle θ, $\tan\theta = \frac{\text{opp}}{\text{adj}}$

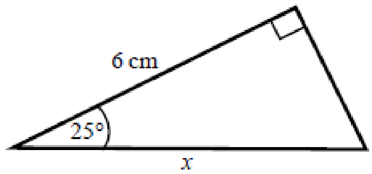
- If the lengths of two sides of a right-angled triangle are given, you can find a missing angle using the inverse trigonometric functions: $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

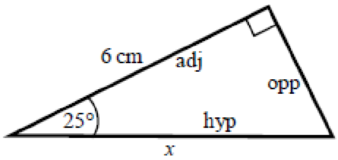
- The sine, cosine and tangent of some angles may be written exactly.

	0	**30°**	**45°**	**60°**	**90°**
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

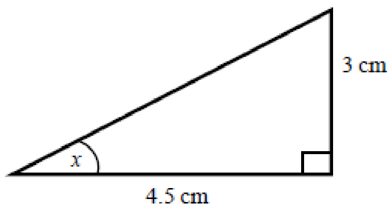
Examples

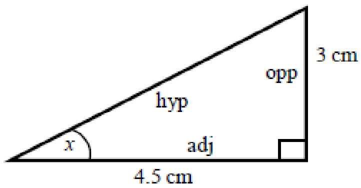
Example 1 Calculate the length of side x.
Give your answer correct to 3 significant figures.



	**1** Always start by labelling the sides.
$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	**2** You are given the adjacent and the hypotenuse so use the cosine ratio.
$\cos 25° = \frac{6}{x}$	**3** Substitute the sides and angle into the cosine ratio.
$x = \frac{6}{\cos 25°}$	**4** Rearrange to make x the subject.
$x = 6.620\,267\,5...$	**5** Use your calculator to work out $6 \div \cos 25°$.
$x = 6.62$ cm	**6** Round your answer to 3 significant figures and write the units in your answer.

Example 2 Calculate the size of angle x.
Give your answer correct to 3 significant figures.





$\tan\theta = \frac{\text{opp}}{\text{adj}}$

$\tan x = \frac{3}{4.5}$

$x = \tan^{-1}\left(\frac{3}{4.5}\right)$

$x = 33.690\,067\,5...$

$x = 33.7°$

1 Always start by labelling the sides.

2 You are given the opposite and the adjacent so use the tangent ratio.

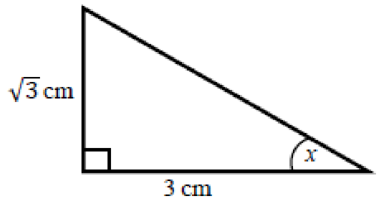
3 Substitute the sides and angle into the tangent ratio.

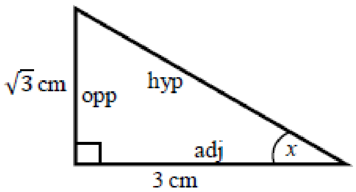
4 Use $\tan^{-1}$ to find the angle.

5 Use your calculator to work out $\tan^{-1}(3 \div 4.5)$.

6 Round your answer to 3 significant figures and write the units in your answer.

Example 3 Calculate the exact size of angle *x*.





$\tan\theta = \frac{\text{opp}}{\text{adj}}$

$\tan x = \frac{\sqrt{3}}{3}$

$x = 30°$

1 Always start by labelling the sides.

2 You are given the opposite and the adjacent so use the tangent ratio.

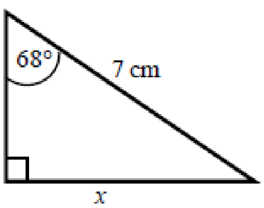
3 Substitute the sides and angle into the tangent ratio.

4 Use the table from the key points to find the angle.

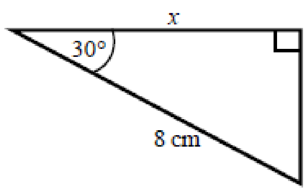
Practice

1 Calculate the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

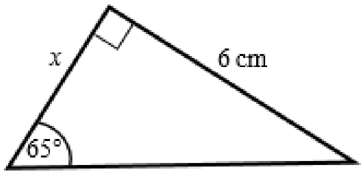
a



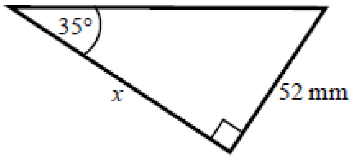
b



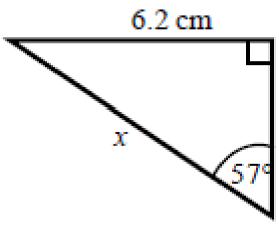
c



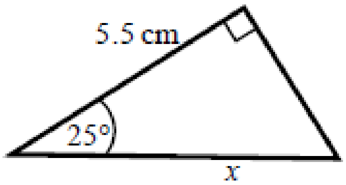
d



e

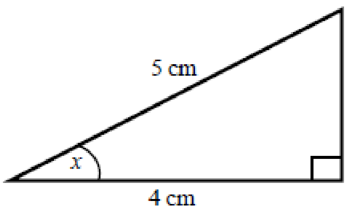


f

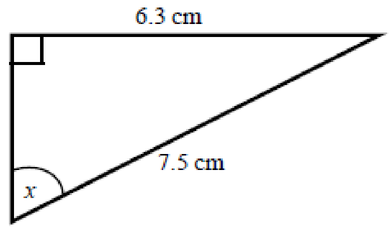


2 Calculate the size of angle x in each triangle.
Give your answers correct to 1 decimal place.

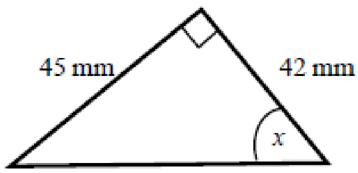
a



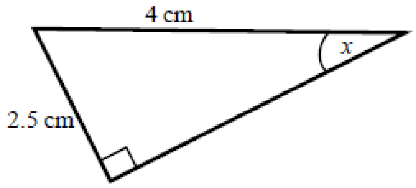
b



c

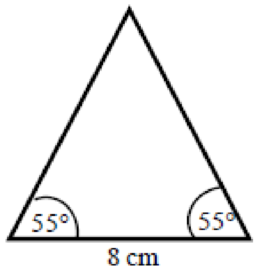


d



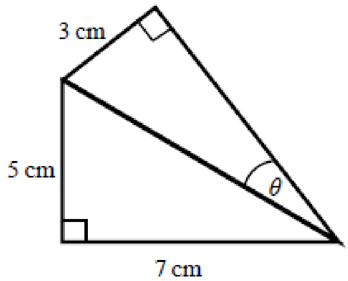
3 Work out the height of the isosceles triangle.
Give your answer correct to 3 significant figures.

> **Hint:**
>
> Split the triangle into two right-angled triangles.



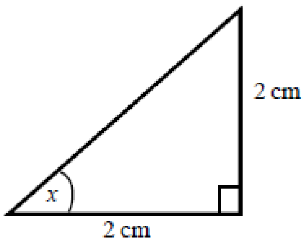
4 Calculate the size of angle θ.
Give your answer correct to 1 decimal place.

> **Hint:**
>
> First work out the length of the common side to both triangles, leaving your answer in surd form.

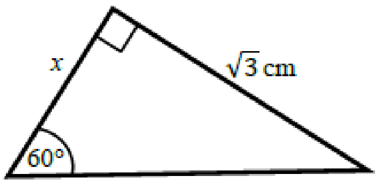


5 Find the exact value of x in each triangle.

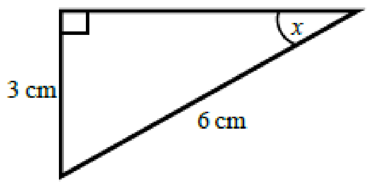
a



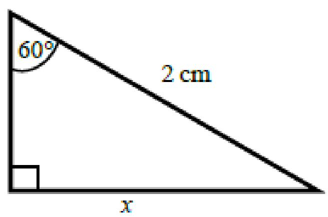
b



c



d



By the end of this task you will:

- have gained knowledge and understanding of the cosine rule
- be able to find a missing side
- be able to find a missing angle

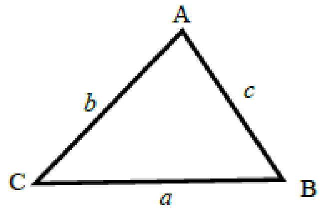
Task 11:

The cosine rule

A LEVEL LINKS
Trigonometric ratios and graphs

Key points

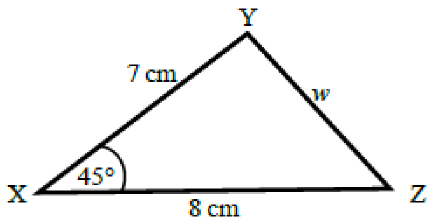
- a is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.

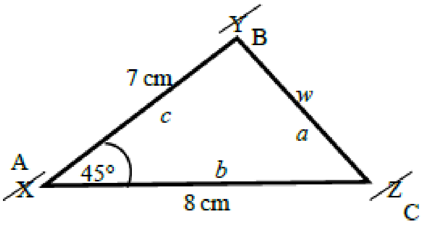


- You can use the cosine rule to find the length of a side when two sides and the included angle are given.
- To calculate an unknown side use the formula $a^2 = b^2 + c^2 - 2bc\cos A$.
- Alternatively, you can use the cosine rule to find an unknown angle if the lengths of all three sides are given.
- To calculate an unknown angle use the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Examples

Example 4 Work out the length of side w.
Give your answer correct to 3 significant figures.





$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$w^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 45°$$

$w^2 = 33.804\,040\,51...$

$w = \sqrt{33.80404051}$

$w = 5.81$ cm

1 Always start by labelling the angles and sides.

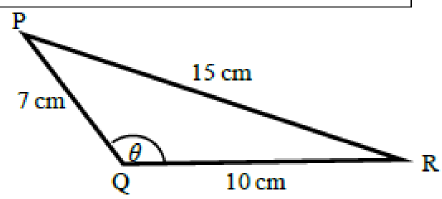
2 Write the cosine rule to find the side.

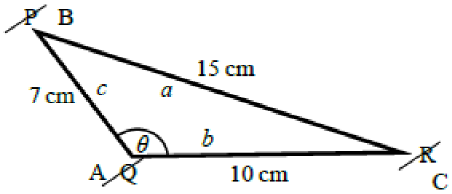
3 Substitute the values a, b and A into the formula.

4 Use a calculator to find w^2 and then w.

5 Round your final answer to 3 significant figures and write the units in your answer.

Example 5 Work out the size of angle θ.
Give your answer correct to 1 decimal place.





$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\theta = \frac{10^2 + 7^2 - 15^2}{2 \times 10 \times 7}$$

$$\cos\theta = \frac{-76}{140}$$

$\theta = 122.878\,349...$

$\theta = 122.9°$

1 Always start by labelling the angles and sides.

2 Write the cosine rule to find the angle.

3 Substitute the values a, b and c into the formula.

4 Use $\cos^{-1}$ to find the angle.

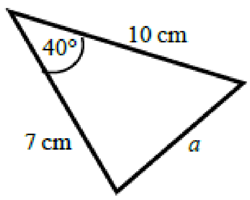
5 Use your calculator to work out $\cos^{-1}(-76 \div 140)$.

6 Round your answer to 1 decimal place and write the units in your answer.

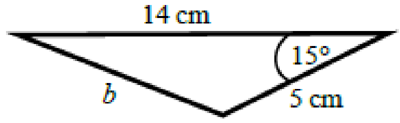
Practice

6 Work out the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

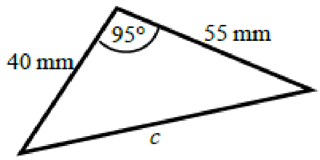
a



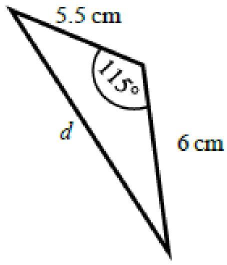
b



c

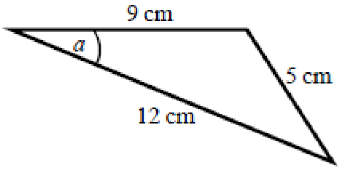


d

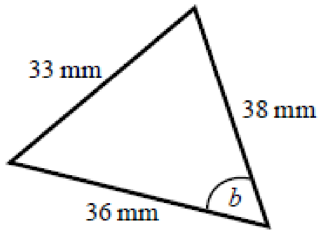


7 Calculate the angles labelled θ in each triangle.
Give your answer correct to 1 decimal place.

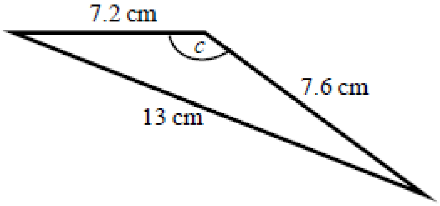
a



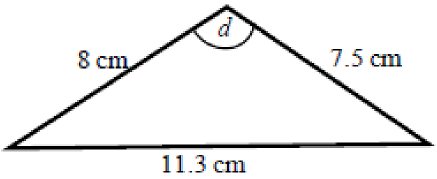
b



c

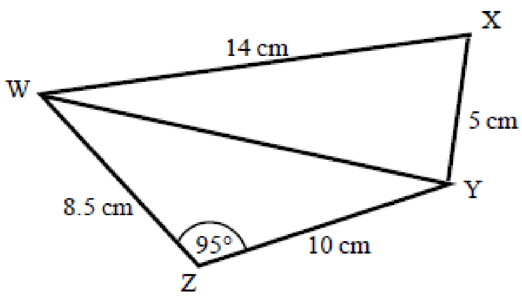


d



8 **a** Work out the length of WY.
Give your answer correct to
3 significant figures.

b Work out the size of angle WXY.
Give your answer correct to
1 decimal place.



By the end of this task you will:

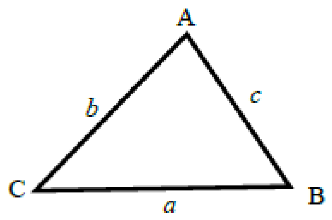
- have gained knowledge and understanding of the sine rule
- be able to find a missing side
- be able to find a missing angle

Task 12:

The sine rule

A LEVEL LINKS
Trigonometric ratios and graphs

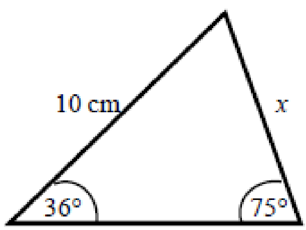
Key points

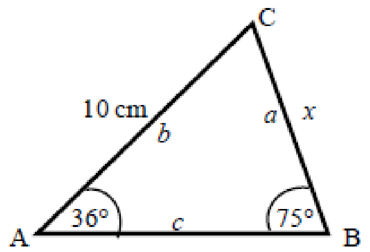


- *a* is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.
- You can use the sine rule to find the length of a side when its opposite angle and another opposite side and angle are given.
- To calculate an unknown side use the formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.
- Alternatively, you can use the sine rule to find an unknown angle if the opposite side and another opposite side and angle are given.
- To calculate an unknown angle use the formula $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Examples

Example 6 Work out the length of side *x*.
Give your answer correct to 3 significant figures.





$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 36°} = \frac{10}{\sin 75°}$$

$$x = \frac{10 \times \sin 36°}{\sin 75°}$$

$x = 6.09$ cm

1 Always start by labelling the angles and sides.

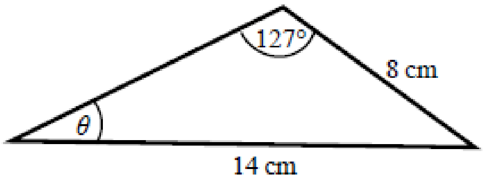
2 Write the sine rule to find the side.

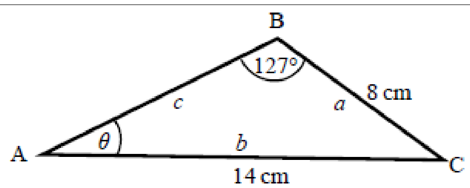
3 Substitute the values a, b, A and B into the formula.

4 Rearrange to make x the subject.

5 Round your answer to 3 significant figures and write the units in your answer.

Example 7 Work out the size of angle θ.
Give your answer correct to 1 decimal place.





$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{8} = \frac{\sin 127°}{14}$$

$$\sin \theta = \frac{8 \times \sin 127°}{14}$$

$\theta = 27.2°$

1 Always start by labelling the angles and sides.

2 Write the sine rule to find the angle.

3 Substitute the values a, b, A and B into the formula.

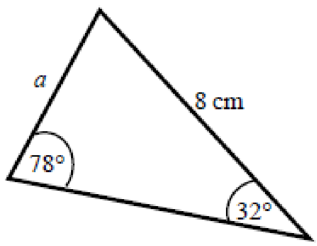
4 Rearrange to make $\sin \theta$ the subject.

5 Use $\sin^{-1}$ to find the angle. Round your answer to 1 decimal place and write the units in your answer.

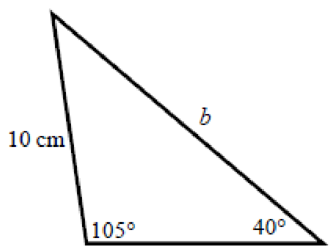
Practice

9 Find the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

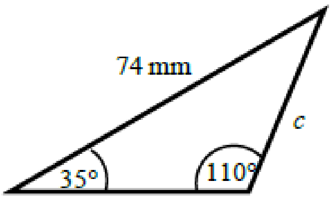
a



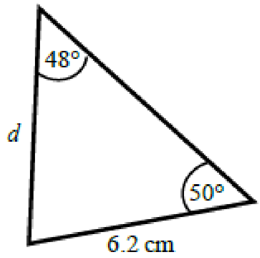
b



c

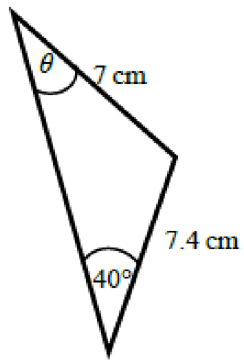


d

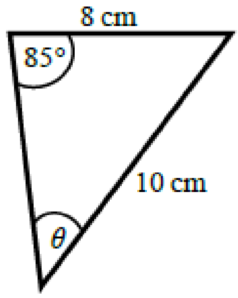


10 Calculate the angles labelled θ in each triangle.
Give your answer correct to 1 decimal place.

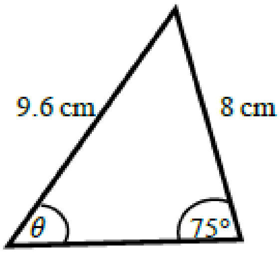
a



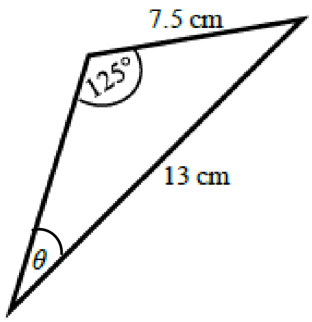
b



c

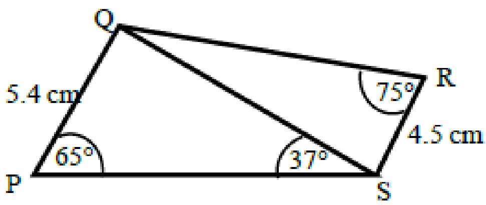


d



11 **a** Work out the length of QS.
Give your answer correct to 3 significant figures.

b Work out the size of angle RQS.
Give your answer correct to 1 decimal place.



By the end of this task you will:

- have gained knowledge and understanding of using trig to work out the area of a triangle
- be able to work out the area of a triangle

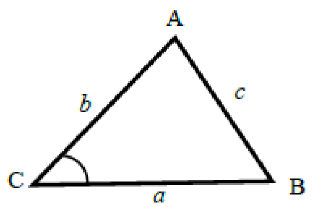
<u>**Task 13:**</u>

Areas of triangles

A LEVEL LINKS
Trigonometric ratios and graphs

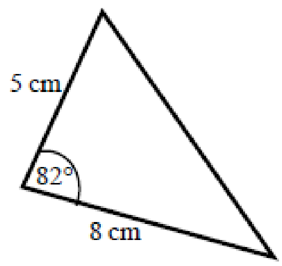
Key points

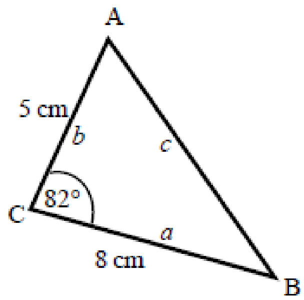
- a is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.
- The area of the triangle is $\frac{1}{2}ab\sin C$.



Examples

Example 8 Find the area of the triangle.

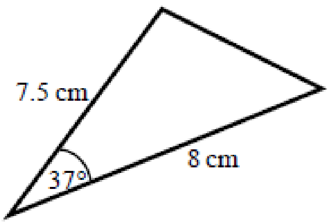


	**1** Always start by labelling the sides and angles of the triangle.
Area = $\frac{1}{2}ab\sin C$	**2** State the formula for the area of a triangle.
Area = $\frac{1}{2}\times 8\times 5\times \sin 82°$	**3** Substitute the values of a, b and C into the formula for the area of a triangle.
Area = 19.805 361...	**4** Use a calculator to find the area.
Area = 19.8 cm^2	**5** Round your answer to 3 significant figures and write the units in your answer.

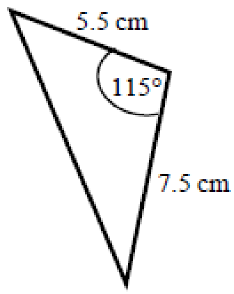
Practice

12 Work out the area of each triangle.
Give your answers correct to 3 significant figures.

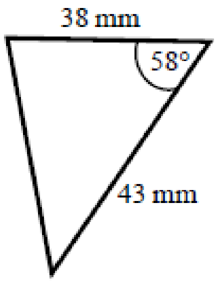
a



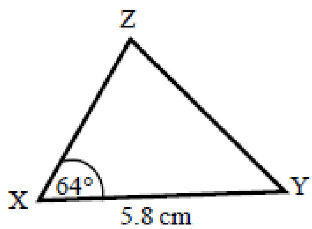
b



c



13 The area of triangle XYZ is 13.3 cm^2.
Work out the length of XZ.



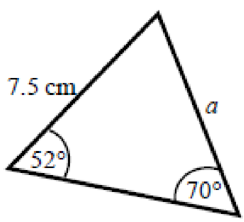
> **Hint:**
>
> Rearrange the formula to make a side the subject.

Extend

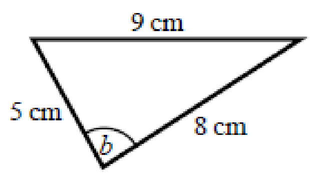
14 Find the size of each lettered angle or side.
Give your answers correct to 3 significant figures.

> **Hint:**
>
> For each one, decide whether to use the cosine or sine rule.

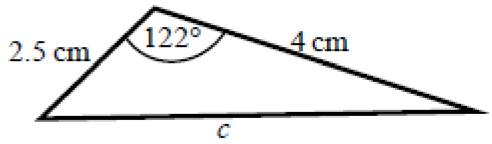
a



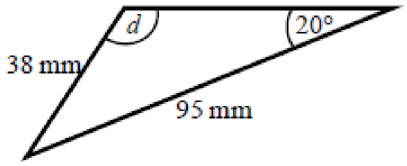
b



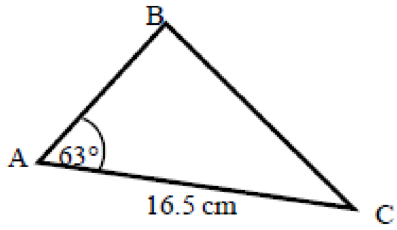
c



d



15 The area of triangle ABC is 86.7 cm^2.
Work out the length of BC.
Give your answer correct to 3 significant figures.



Answers

1	**a**	6.49 cm	**b**	6.93 cm	**c**	2.80 cm		
	d	74.3 mm	**e**	7.39 cm	**f**	6.07 cm		
2	**a**	36.9°	**b**	57.1°	**c**	47.0°	**d**	38.7°
3	5.71 cm							
4	20.4°							
5	**a**	45°	**b**	1 cm	**c**	30°	**d**	$\sqrt{3}$ cm
6	**a**	6.46 cm	**b**	9.26 cm	**c**	70.8 mm	**d**	9.70 cm
7	**a**	22.2°	**b**	52.9°	**c**	122.9°	**d**	93.6°
8	**a**	13.7 cm	**b**	76.0°				
9	**a**	4.33 cm	**b**	15.0 cm	**c**	45.2 mm	**d**	6.39 cm
10	**a**	42.8°	**b**	52.8°	**c**	53.6°	**d**	28.2°
11	**a**	8.13 cm	**b**	32.3°				
12	**a**	18.1 cm^2	**b**	18.7 cm^2	**c**	693 mm^2		
13	5.10 cm							
14	**a**	6.29 cm	**b**	84.3°	**c**	5.73 cm	**d**	58.8°
15	15.3 cm							

By the end of this task you will:

- ❑ have gained knowledge and understanding of rearranging equations
- ❑ be able to change the subject of an equation

Task 14:

Rearranging equations

A LEVEL LINKS
Definition, differentiating polynomials, second derivatives

Key points

- To change the subject of a formula, get the terms containing the subject on one side and everything else on the other side.
- You may need to factorise the terms containing the new subject.

Examples

Example 1 Make t the subject of the formula $v = u + at$.

$v = u + at$ $v - u = at$ $t = \frac{v-u}{a}$	**1** Get the terms containing t on one side and everything else on the other side. **2** Divide throughout by a.

Example 2 Make t the subject of the formula $r = 2t - \pi t$.

$r = 2t - \pi t$ $r = t(2 - \pi)$ $t = \frac{r}{2-\pi}$	**1** All the terms containing t are already on one side and everything else is on the other side. **2** Factorise as t is a common factor. **3** Divide throughout by $2 - \pi$.

Example 3 Make t the subject of the formula $\frac{t+r}{5} = \frac{3t}{2}$.

$\frac{t+r}{5} = \frac{3t}{2}$ $2t + 2r = 15t$ $2r = 13t$ $t = \frac{2r}{13}$	**1** Remove the fractions first by multiplying throughout by 10. **2** Get the terms containing t on one side and everything else on the other side and simplify. **3** Divide throughout by 13.

Example 4 Make t the subject of the formula $r = \dfrac{3t+5}{t-1}$.

$r = \dfrac{3t+5}{t-1}$	**1** Remove the fraction first by multiplying throughout by $t - 1$.
$r(t-1) = 3t+5$	**2** Expand the brackets.
$rt - r = 3t + 5$	**3** Get the terms containing t on one side and everything else on the other side.
$rt - 3t = 5 + r$	**4** Factorise the LHS as t is a common factor.
$t(r-3) = 5 + r$	**5** Divide throughout by $r - 3$.
$t = \dfrac{5+r}{r-3}$	

Practice

Change the subject of each formula to the letter given in the brackets.

1 $C = \pi d$ [d]

2 $P = 2l + 2w$ [w]

3 $D = \dfrac{S}{T}$ [T]

4 $p = \dfrac{q-r}{t}$ [t]

5 $u = at - \frac{1}{2}t$ [t]

6 $V = ax + 4x$ [x]

7 $\dfrac{y-7x}{2} = \dfrac{7-2y}{3}$ [y]

8 $x = \dfrac{2a-1}{3-a}$ [a]

9 $x = \dfrac{b-c}{d}$ [d]

10 $h = \dfrac{7g-9}{2+g}$ [g]

11 $e(9 + x) = 2e + 1$ [e]

12 $y = \dfrac{2x+3}{4-x}$ [x]

13 Make r the subject of the following formulae.

a $A = \pi r^2$ **b** $V = \frac{4}{3}\pi r^3$ **c** $P = \pi r + 2r$ **d** $V = \frac{2}{3}\pi r^2 h$

14 Make x the subject of the following formulae.

a $\dfrac{xy}{z} = \dfrac{ab}{cd}$ **b** $\dfrac{4\pi cx}{d} = \dfrac{3z}{py^2}$

15 Make sin B the subject of the formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

16 Make cos B the subject of the formula $b^2 = a^2 + c^2 - 2ac \cos B$.

Extend

17 Make x the subject of the following equations.

a $\frac{p}{q}(sx+t)=x-1$ **b** $\frac{p}{q}(ax+2y)=\frac{3p}{q^2}(x-y)$

Answers

1 $d = \frac{C}{\pi}$

2 $w = \frac{P - 2l}{2}$

3 $T = \frac{S}{D}$

4 $t = \frac{q - r}{p}$

5 $t = \frac{2u}{2a - 1}$

6 $x = \frac{V}{a + 4}$

7 $y = 2 + 3x$

8 $a = \frac{3x + 1}{x + 2}$

9 $d = \frac{b - c}{x}$

10 $g = \frac{2h + 9}{7 - h}$

11 $e = \frac{1}{x + 7}$

12 $x = \frac{4y - 3}{2 + y}$

13 **a** $r = \sqrt{\frac{A}{\pi}}$ **b** $r = \sqrt[3]{\frac{3V}{4\pi}}$

c $r = \frac{P}{\pi + 2}$ **d** $r = \sqrt{\frac{3V}{2\pi h}}$

14 **a** $x = \frac{abz}{cdy}$ **b** $x = \frac{3dz}{4\pi cpy^2}$

15 $\sin B = \frac{b \sin A}{a}$

16 $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

17 **a** $x = \frac{q + pt}{q - ps}$ **b** $x = \frac{3py + 2pqy}{3p - apq} = \frac{y(3 + 2q)}{3 - aq}$

Wider reading:

Below are some articles and videos to view.
These are all going to extend your understanding of maths in the real world.

1. **Follow the 'WATCH, THINK, DIG DEEPER, DISCUSS'**
   The Wizard standoff riddle.
   https://ed.ted.com/lessons/can-you-solve-the-wizard-standoff-riddle-daniel-finkel

2. **Follow the 'WATCH, THINK, DIG DEEPER, DISCUSS'**
   Solve the false positive riddle.
   https://ed.ted.com/lessons/can-you-solve-the-false-positive-riddle-alex-gendler

3. **Read the notes on the page and carry out the algebraic investigation. Complete the worksheet included.**
   https://www.teachmathematics.net/page/7566/oxo

4. **Create a PINTREST board with images of maths in nature. Investigate the maths behind some of the images you have found.**

5. **Maths Magic.**
   Can you create your own version of the problem? Investigate other magic tricks which are based around maths.
   https://nrich.maths.org/1051

6. **Golden Ratio Day**
   Golden ratio day is 1st June 2018. Investigate the golden ratio and its history.
   https://www.teachengineering.org/activities/view/nyu_phi_activity1
   https://www.quora.com/How-is-the-golden-ratio-useful-to-students
   Find more articles on this and create a poster all about the golden ratio.

7. **Complete module 1- Advanced Problem Solving**
   https://nrich.maths.org/10209

Other reading

- Devlin, K. (2004) The millennium problems: the seven greatest, unsolved mathematical problems of our time.
- Dunham, W. (1991) Journey through genius: the greatest theorems of mathematics.

- Du Sautoy, M. (2003) The music of the primes : why an unsolved problem in mathematics matters.
- Enzensberger, H. (2008) The number devil (very accessible, fun read).
- Frankel, L. (1997) Numbers: the universal language (very engaging and accessible read).
- Use the weblinks on Integral.
- http://nrich.maths.org/secondary-upper